AN EXPLICIT ISOMORPHISM BETWEEN FLOER HOMOLOGY AND QUANTUM HOMOLOGY

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ABSTRACT. We use Liu-Tian's virtual moduli cycle methods to construct detailedly the explicit isomorphism between Floer homology and quantum homology for any closed symplectic manifold that was first outlined by Piunikhin, Salamon and Schwarz for the case of the semi-positive symplectic manifolds.

1. INTRODUCTION

1.1. **Background and motivation.** It is one of the exciting mathematics achievements in the last few years that the Floer and quantum homologies were established for all closed symplectic manifolds (see [FuO, LiT, LiuT1, R, Sie] and [HS2]). (Less general versions had been obtained before by Floer [F], Hofer-Salamon [HS1], Ruan-Tian [RT1] and McDuff-Salamon [McS].) The purpose of this paper is to construct detailedly an explicit ring isomorphism between them. Such an isomorphism was first outlined by Piunikhin, Salamon and Schwarz in the semi-positive case [PSSc]. Our argument is based on Liu-Tian's virtual cycle methods in [LiuT1]-[LiuT3]. The isomorphism is necessary and convenient for studies of some symplectic topology problems, e.x., the topology and geometry of the group $\mathrm{Ham}(M,\omega)$ of Hamiltonian automorphisms of a symplectic manifold (M,ω). Let $\widetilde{G}$ be the group of pairs $(g,\tilde{g})$ consisting of a smooth loop $g:S^1\to\mathrm{Ham}(M,\omega)$ such that $g(0)=Id$ and a lift $\tilde{g}:\tilde{\mathcal{L}}(M)\to\tilde{\mathcal{L}}(M)$ of the action of g to a covering of the space $\mathcal{L}(M)$ of contractible loops in M (see §1.2 below). In a beautiful paper [Se] by Seidel, for every pair $(g,\tilde{g})\in\widetilde{G}$ there is assigned an automorphism $HF_*(g,\tilde{g})$ of the Floer homology $HF_*(M,\omega)$; he constructed a homomorphism q from $\widetilde{G}$ to the group $QH_*(M,\omega)^\times$ of homogeneous even-dimensional invertible elements of the quantum homology ring $QH_*(M,\omega)$ and proved his main result:

$$HF_*(g,\tilde{g})(b)=\Psi^+(q(g,\tilde{g}))*_{PP}b$$

for any $(g,\tilde{g})\in\widetilde{G}$ and $b\in HF_*(M,\omega)$. Here $*_{PP}$ and Ψ^+ are the 'pair-of-pants' product in $HF_*(M,\omega)$ and the canonical isomorphism from $QH_*(M,\omega)$ to $HF_*(M,\omega)$ constructed in [PSSc] respectively. A key step in the proof of his

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main result is Theorem 8.2 on the page 1080 of [Se], whose proof was based on the arguments of [PSSc].

Schwarz [Sch3] defined and analyzed a bi-invariant metric on $\mathrm{Ham}(M,\omega)$ with the construction of such an explicit isomorphism on a closed symplectic manifold (M,ω) with $c_1|_{\pi_2(M)} = \omega|_{\pi_2(M)} = 0$. Recently, Oh [Oh] obtained the corresponding results on arbitrary closed symplectic manifolds. As pointed out in $\S 5.3$ of [Oh] it would seem more natural to use the Piunikhin-Salamon-Schwarz map in the definition of his mini-max value function ρ. Entov [En] studied the relations between the K-area for Hamiltonian fibrations with a strongly semi-positive typical fiber (M,ω) over a surface with boundary and the Hofer geometry on the group $\mathrm{Ham}(M,\omega)$. Such a ring isomorphism was used to obtain a key estimate in his work.

Since these applications used the Piunikhin-Salamon-Schwarz isomorphism in [PSSc] our detailed generalization to arbitrary closed symplectic manifolds may be used to generalize their results to the desired forms more directly and conveniently. Moreover the method to construct the ring isomorphism has actually more uses than the isomorphism itself because not only the isomorphism itself but also the map of Piunikhin-Salamon-Schwarz'type in the chain level were used in some applications. The construction of another ring isomorphism was given by Liu-Tian [LiuT3] (a less general version was announced before by Ruan-Tian [RT2]). Without doubt different construction methods of the ring isomorphisms between Floer homology and quantum homology have respective advantages in the studies of different symplectic topology problems.

1.2. **Outline and the main result.** For a smooth non-degenerate time-dependent function $H : M\times\mathbb{R}/\mathbb{Z}\to\mathbb{R}$ one may associate a family of the Hamiltonian vector fields X_{H_t} by $\omega(X_{H_t},\cdot) = -dH_t$ for $t\in\mathbb{R}$ and $H_t(\cdot) = H(t,\cdot)$. Let $\mathcal{P}(H)$ be the set of all contractible 1-periodic solutions of the Hamiltonian differential equation: $\dot{x}(t) = X_{H_t}(x(t))$. Denote by $\mathcal{J}(M,\omega)$ the space of all almost complex structures compatible with ω. It determines a unique the first Chern class $c_1 = c_1(TM,J)\in H^2(M,\mathbb{Z})$ via any $J\in\mathcal{J}(M,\omega)$([Gr]). Let $\phi_{c_1},\phi_\omega : H_2^S(M)\to\mathbb{R}$ be the homomorphisms by evaluations of c_1 and ω respectively. Here $H_2^S(M)$ denotes the image of $\pi_2(M)$ in $H_2(M;\mathbb{Z})$ under the Hurewicz homomorphism modulo torsion. As usual let $\mathcal{L}(M)$ be the set of all contractible loops $x\in C^\infty(S^1,M)$. Consider a pair (x,v) consisting of $x\in\mathcal{L}(M)$ and a disk v bounding x. Such two pairs (x,v) and (y,w) are called equivalent if $x=y$ and ϕ_{c_1},ϕ_ω both take zero value on $v\sharp(-w)$. Denote by $[x,v]$ the equivalence class of a pair (x,v) and by $\widetilde{\mathcal{L}(M)}$ the set of all such equivalence classes. Then the latter is a cover space of $\mathcal{L}(M)$ with the covering transformation group $\Gamma = H_2^S(M)/(\ker\phi_{c_1}\cap\ker\phi_\omega)$. Its action is given by $A\cdot[x,v] = [x, A\sharp v]$ for any $A\in\Gamma$, where $A\sharp v$ is understood as the connected sum of any representative of A in $\pi_2(M)$ with v. In this paper we shall denote $[x,v]$ by $\tilde{x}$ and $[x,v\sharp A]$ by $\tilde{x}\sharp A$ for $A\in\Gamma$ if it is not necessary to point out the bounding disk v and no confusion occurs. Let $\tilde{\mathcal{P}}(H)$ be the lifting of

$\mathcal{P}(H)$ in the space $\widetilde{\mathcal{L}(M)}$. It is exactly the critical set of the functional

$$\mathcal{F}_H : \widetilde{\mathcal{L}(M)} \to \mathbb{R},\ [x,v] \mapsto -\int_D v^*\omega + \int_0^1 H(t,x(t))dt$$

on $\widetilde{\mathcal{L}(M)}$. Given $\tilde{x}^\pm = [x^\pm, v^\pm] \in \tilde{\mathcal{P}}(H)$ denote by

$$\mathcal{M}(\tilde{x}^-, \tilde{x}^+; H, J) \tag{1.1}$$

the space of all connecting trajectories $u : \mathbb{R} \times S^1 \to M$ satisfying the equation $\bar{\partial}_{J,H} u = \partial_s u + J(u)\partial_t u + \nabla H_t(u) = 0$ with boundary conditions $\lim_{s\to\pm\infty} u(s,t) = x^\pm$ and $[x^+, v^- \sharp u] = [x^+, v^+]$. After making a small generic perturbation of H_t outside some small neighborhood of the graph of the elements of $\mathcal{P}(H)$ one may assume that for any two $\tilde{x}^\pm \in \tilde{\mathcal{P}}(H)$ the space $\mathcal{M}(\tilde{x}^-, \tilde{x}^+; H, J)$ is either an empty set or a manifold of dimension $\mu(\tilde{x}^-) - \mu(\tilde{x}^+)$ ([F]). Here $\mu(\tilde{x})$ is the Conley-Zehnder index of $\tilde{x}$ ([SZ]). Denote by $\tilde{\mathcal{P}}_k(H) := \{\tilde{x} \in \tilde{\mathcal{P}}(H) \,|\, \mu(\tilde{x}) = k\}$. Consider the chain complex whose k-th chain group $C_k(H, J; \mathbb{Q})$ consists of all formal sums $\sum \xi_{\tilde{x}} \cdot \tilde{x}$ with $\xi_{\tilde{x}} \in \mathbb{Q}$ and $\tilde{x} \in \tilde{\mathcal{P}}_k(H)$ such that the set $\{\tilde{x} \in \tilde{\mathcal{P}}_k(H) \,|\, \xi_{\tilde{x}} \neq 0,\ \mathcal{F}_H(\tilde{x}) > c\}$ is finite for any $c \in \mathbb{R}$. Then $C_*(H, J; \mathbb{Q}) = \oplus_k C_k(H, J; \mathbb{Q})$ is a graded $\mathbb{Q}$-space of infinite dimension. However its dimension as a module over the Novikov ring $\Lambda_\omega = \Lambda_\omega(\mathbb{Q})$ is finite. Here $\Lambda_\omega(\mathbb{Q})$ is the collection of all formal sums $\lambda = \sum \lambda_A \cdot e^A$ with $\lambda_A \in \mathbb{Q}$ such that the set $\{A \in \Gamma \,|\, \lambda_A \neq 0, \omega(A) < c\}$ is finite for any $c \in \mathbb{R}$. Its action on $C_*(H, J; \mathbb{Q})$ is defined by $(\lambda * \xi)_{\tilde{x}} = \sum_{A\in\Gamma} \lambda_A \xi_{(-A)\cdot\tilde{x}}$ and the rank of $C_*(H, J; \mathbb{Q})$ over Λ_ω is equal to $\sharp\mathcal{P}(H)$. When (M, ω) is a monotone symplectic manifold and $\mu(\tilde{x}^-) - \mu(\tilde{x}^+) = 1$ Floer proved that the manifold in (1.1) is compact and thus first established his Floer homology theory [F]. Later on his arguments were generalized to the semi-positive case by Hofer-Salamon [HS1] and Ono [O], and the case of the product of semi-positive symplectic manifolds by author [Lu1]. Note that the space in (1.1) is not compact in general case. It is this noncompactness that impedes the establishment of Floer homology theory on all closed symplectic manifolds. Let us outline Liu-Tian's method to overcome this difficulty since we shall choose their method to realize our program. Replacing the space in (1.1) they considered $\overline{\mathcal{M}}(\tilde{x}^-, \tilde{x}^+; J, H)$ the space of the equivalence classes of all (J, H)-stable trajectories from $\tilde{x}^-$ to $\tilde{x}^+$(cf. Def.2.1), and used it to construct a suitable relative virtual moduli cycle $C(\overline{\mathcal{M}}^\nu(\tilde{x}^-, \tilde{x}^+))$ of dimension $\mu(\tilde{x}^-) - \mu(\tilde{x}^+) - 1$(see §2.2.) If $\mu(\tilde{x}^-) - \mu(\tilde{x}^+) = 1$ the virtual moduli cycle may determine a rational number $\sharp\big(C(\overline{\mathcal{M}}^\nu(\tilde{x}^-, \tilde{x}^+))\big)$ (see [LiuT1]). Then for each $\xi = \sum_{\tilde{x}} \xi_{\tilde{x}} \tilde{x}$ in $C_k(H, J; \mathbb{Q})$ they defined

$$\partial_k^F \xi = \sum_{\mu(\tilde{y})=k-1} \Bigg[\sum_{\mu(\tilde{x})=k} \sharp\big(C(\overline{\mathcal{M}}^\nu(\tilde{x}, \tilde{y}))\big) \cdot \xi_{\tilde{x}} \Bigg] \tilde{y}. \tag{1.2}$$

and proved it to be indeed a boundary operator. Let $HF_*(M, \omega; H, J, \nu; \mathbb{Q})$ be the homology of the above chain complex. Using Floer's deformation ideas they

proved that this homology is invariant under deformations and also isomorphic to $H_*(M,\mathbb{Q})\otimes\Lambda_\omega$, i. e., the quantum homology of M.

For the construction of our isomorphism let us firstly fix a Morse function h_0 on M and a small open neighborhood $\mathcal{O}(h_0)$ of it in $C^\infty(M)$ such that

(i) for some $\epsilon>0$ any two different points a and b of $\mathrm{Crit}(h_0)=\{a_1,\cdots,a_m\}$ have a distance $d(a,b)>4\epsilon$ with respect to some distance d on M;

(ii) any two different critical points a^h and b^h of $h\in\mathcal{O}(h_0)$ have a distance $d(a^h,b^h)>3\epsilon$;

(iii) each $h\in\mathcal{O}(h_0)$ has a unique critical point a_i^h in each ball $B_d(a_i,\epsilon)=\{c\in M\,|\,d(a_i,c)<\epsilon\}$ and no other critical points (hence $\sharp\mathrm{Crit}(h)=m$);

(iv) for any $h\in\mathcal{O}(h_0)$ the Morse index $\mu(a_i^h)=\mu(a_i)$, $i=1,\cdots,m$;

(v) the function

$$\mathcal{O}(h_0)\to B_d(a_1,\epsilon)\times\cdots\times B_d(a_m,\epsilon),\ h\mapsto(a_1^h,\cdots,a_m^h)$$

is a smooth surjective map.

Take $h\in\mathcal{O}(h_0)$ and a Riemannian metric g on M such that (h,g) is a Morse-Smale pair. As in [PSSc] we may use the solutions $\gamma:\mathbb{R}\to M$ of

$$\dot\gamma(s)=-\nabla^g h(\gamma(s)) \tag{1.3}$$

to construct a chain complex expression of the quantum homology $H_*(M,\mathbb{Q})\otimes\Lambda_\omega$ as follows: For every integer k let us denote by

$$QC_k(M,\omega;h,g;\mathbb{Q}) \tag{1.4}$$

the set of all formal sums $\zeta=\sum_{\mu(\langle a,A\rangle)=k}\zeta_{(a,A)}a\oplus A$ such that $\{(a,A)\in(\mathrm{Crit}(h)\times\Gamma)_k\,|\,\zeta(a,A)\neq0,\ h(a)-\phi_\omega(A)>c\}$ is a finite set for all $c\in\mathbb{R}$. Here $(\mathrm{Crit}(h)\times\Gamma)_k:=\{(a,A)\in\mathrm{Crit}(h)\times\Gamma\mid\mu(\langle a,A\rangle):=\mu(a)-2c_1(A)=k\}$. The action of Λ_ω on $QC_*(M,\omega;h,g;\mathbb{Q})$ is given by

$$\lambda\star\zeta=\sum_{(b,B)\in\mathrm{Crit}(h)\times\Gamma}\left(\sum_{2c_1(A)=\mu(\langle b,B\rangle)-k}\lambda_{\alpha\oplus A}\zeta_{\langle(-\alpha)\cdot b,B-A\rangle}\right)\langle b,B\rangle. \tag{1.5}$$

The boundary operator $\partial_k^Q:QC_k(M,\omega;h,g;\mathbb{Q})\to QC_{k-1}(M,\omega;h,g;\mathbb{Q})$ is given by

$$\partial_k^Q(\langle a,A\rangle)=\textstyle\sum_{\mu(b)=\mu(a)-1}n(a,b)\langle b,A\rangle, \tag{1.6}$$

where $n(a,b)$ is the oriented number of the solutions of (1.3) from a to b. It is easily checked that $\lambda\star\zeta\in QC_*(M,\omega;h,g;\mathbb{Q})$ and that the latter is a graded vector space over $\Lambda_\omega(\mathbb{Q})$ according to the multiplication defined in (1.5), and that ∂^Q is a boundary operator and also $\Lambda_\omega(\mathbb{Q})$-linear with respect to the multiplication. Consequently, $(QC_*(M,\omega;h,g;\mathbb{Q}),\partial^Q)$ is a chain complex. Let us denote its homology by

$$QH_*(h,g;\mathbb{Q}):=H_*(QC_*(M,\omega;h,g;\mathbb{Q}),\ \ \partial^Q).$$

It is easy to derive from [Sch4] that there exists an explicit graded Λ_ω-module isomorphism between $QH_*(h,g;\mathbb{Q})$ and $H_*(M;\mathbb{Q})\otimes\Lambda_\omega$([Sch2]).

To construct an explicit Λ_ω-module isomorphism between $QH_*(h,g;\mathbb{Q})$ and $HF_*(M,\omega;H,J,\nu;\mathbb{Q})$ we shall associated two rational numbers $n_+^{\nu^+}(a,\tilde{x}\sharp(-A))$ and $n_-^{\nu^-}(a,\tilde{x}\sharp(-A))$ in (3.2) to both $\langle a,A\rangle\in\mathrm{Crit}(h)\times\Gamma$ and $\tilde{x}\in\tilde{\mathcal{P}}(H)$, and then mimic [PSSc] to construct formally two maps on the levels of chains

$$\Phi(\langle a,A\rangle)=\sum\nolimits_{\mu(\tilde{x})=\mu(\langle a,A\rangle)} n_+^{\nu^+}(a,\tilde{x}\sharp(-A))\tilde{x}, \tag{1.7}$$

$$\Psi(\tilde{x})=\sum\nolimits_{\mu(\langle a,A\rangle)=\mu(\tilde{x})} n_-^{\nu^-}(a,\tilde{x}\sharp(-A))\langle a,A\rangle. \tag{1.8}$$

Indeed, in Remark 3.7 we shall show that Φ and Ψ are Λ_ω-module chain homomorphisms from $QC_*(M,\omega;h,g;\mathbb{Q})$ to $C_*(H,J,\nu;\mathbb{Q})$ and from $C_*(H,J,\nu;\mathbb{Q})$ to $QC_*(M,\omega;h,g;\mathbb{Q})$ respectively. Our main result is:

Theorem 1.1. *Φ induces a Λ_ω-module isomorphism*

$$\Phi_*:QH_*(h,g;\mathbb{Q})\to HF_*(M,\omega;H,J,\nu;\mathbb{Q})$$

with an inverse Ψ_*.

Remark 1.2. As concluding remarks we point out that Theorems 3.1, 3.7 and 5.1 in [PSSc] may easily be extended to any closed symplectic manifold. Such an extension of Theorem 3.1 was actually carried out in [LiuT3]. Following the lines in [PSSc] and combing the methods in [LiuT3] with ones in this paper we easily complete the extension of Theorem 5.1 in [PSSc] to arbitrary closed symplectic manifolds (in fact, a long exercise). As a consequence we get that the isomorphism in Theorem 1.1 is also the ring isomorphism.

Section 2 introduces the moduli spaces of stable disks and constructs the virtual moduli cycles associated with them such that they are compatible with Liu-Tian's relative ones. Section 3 deals with the intersections of these virtual moduli cycles with stable and unstable manifolds. The proof of Theorem 1.1 is completed in §4.

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2. The disk solution spaces and virtual moduli cycles

2.1. **Moduli space of stable disks.** We begin with the disk solution spaces introduced in [PSSc]. For $J\in\mathcal{J}(M,\omega)$ and $[x,v]\in\tilde{\mathcal{P}}(H)$ let $\mathcal{M}_+([x,v];H,J)$ be the set of all smooth maps $u:\mathbb{R}\times S^1\to M$ such that

$$\partial_s u(s,t)+J(u)(\partial_t u-\beta_+(s)X_H(t,u))=0, \tag{2.1}$$

$$u(+\infty)=x\quad\text{and}\quad E_+(u):=\int_{-\infty}^{\infty}\int_0^1|\partial_s u|_{g_J}^2dsdt<+\infty,$$

(2.2) $u\sharp(-v):S^2\to M$ represents a torsion homology class in $H_2(M;\mathbb{Z})$.

Here a smooth cut-off function $\beta_+ : \mathbb{R} \to [0,1]$ is given by

$$\beta_+(s) = \begin{cases} 0 & \text{as } s \le 0 \\ 1 & \text{as } s \ge 1 \end{cases}, \ 2 > \beta'_+(s) > 0, \text{ for } 0 < s < 1.$$

Since $E_+(u) < +\infty$, u extends over the end of $s = -\infty$ by removable singularity theorem. So the connected union $u\sharp(-v)$ in (2.2) is well-defined. If $\mathcal{M}_+([x,v];H,J) \ne \emptyset$ its virtual dimension is $\dim M - \mu([x,v])$.

Correspondingly, for $[x,v] \in \tilde{\mathcal{P}}(H)$ we denote by $\mathcal{M}_-([x,v];H,J)$ the set of all smooth maps $u : \mathbb{R} \times S^1 \to M$ such that

$$\partial_s u(s,t) + J(u)(\partial_t u - \beta_+(-s)X_H(t,u)) = 0, \tag{2.3}$$

$$u(-\infty) = x \quad \text{and} \quad E_-(u) := \int_{-\infty}^{\infty}\int_0^1 |\partial_s u|^2_{g_J} dsdt < +\infty,$$

$$v\sharp u : S^2 \to M \text{ represents a torsion homology class in } H_2(M;\mathbb{Z}). \tag{2.4}$$

Similarly, if $\mathcal{M}_-([x,v];H,J) \ne \emptyset$ its virtual dimension is $\mu([x,v])$. As above we here have extended u over the end of $s = +\infty$. Recall that:

Definition 2.1([LiuT1]). Let $(\Sigma, \underline{l})$ be a semistable $\mathcal{F}$-curve with $z_- = z_1, \cdots, z_{N_p+1} = z_+$, as those double points connecting the principal components(cf. Def.3.1 in [LiuT2]). A continuous map $f : \Sigma \setminus \{z_1, \cdots, z_{N_p+1}\} \to M$ is called a **stable** (J,H)**-map** if there exist $[x_i, v_i] \in \tilde{\mathcal{P}}(H)$, $i = 1, \cdots, N_p+1$, such that:

(1) on each principal component P_i with cylindrical coordinate (s,t) (obtained by the identification $(P_i \setminus \{z_i, z_{i+1}\}; l_i) \equiv (\mathbb{R} \times S^1; \{t = 0\})$), $f_i^P = f|_{P_i - \{z_i, z_{i+1}\}}$ satisfies:
   (i) $\partial_s f_i^P + J(f_i^P)\partial_t f_i^P + \nabla H(f_i^P) = 0$, and
   (ii) $\lim_{s\to -\infty} f_i^P(s,t) = x_i(t)$ and $\lim_{s\to +\infty} f_i^P(s,t) = x_{i+1}(t)$.
(2) The restriction f_j^B of f to each bubble component B_j is J-holomorphic;
(3) $[v_{i+1}] = [v_i] + [f_i^P] + \sum_j [f_{i,j}^B]$ as relative homology class of (M, x_{i+1}), where the domain of $f_{i,j}^B$ may be joined to P_i by a chain of the bubble components not intersecting with other principal components;
(4) All homotopically trivial principal components or homologically trivial bubble components are not free.

The equivalence class of $f = (f, \Sigma, \underline{l})$ is denoted by $\langle f, \Sigma, \underline{l}\rangle$ or simply $\langle f\rangle$(see [LiuT1] for its definition). To compactify the disk solution spaces we introduce:

Definition 2.2. Given a $[x,v] \in \tilde{\mathcal{P}}(H)$ and a semistable $\mathcal{F}$-curve $(\Sigma, \underline{l})$, a continuous map $f : \Sigma \setminus \{z_2, \cdots, z_{N_p+1}\} \to M$ is called a **stable** $(J,H)_+$-**disk** with cap $[x,v]$ if there exist $[x_i, v_i] \in \tilde{\mathcal{P}}(H)$, $i = 2, \cdots, N_p+1$, with $[x,v] = [x_{N_p+1}, v_{N_p+1}]$, such that:

(1) on each principal component P_i $(i > 1)$ with cylindrical coordinate (s,t), $f_i^P = f|_{P_i - \{z_i, z_{i+1}\}}$ satisfies (i) (ii) in Definition 2.1(1), but f_1^P does (2.1) and $f_1^P(+\infty) = x_2$ and $E_+(f_1^P) < +\infty$.

(2) If $N_p > 1$, for each $i > 1$ as relative homology classes of (M, x_{i+1}), $[v_{i+1}] = [v_i] + [f_i^P] + \sum_j [f_{i,j}^B]$ and $[v_2] = [f_1^P] + \sum_j [f_{1,j}^B]$.

(3) All requirements for the bubble components $f_{i,j}^B$ in Definition 2.1(2)(4) still hold. Moreover, all homotopically trivial principal components f_i^P $(i > 1)$ are not free and do not appear in the next way.

Remark 2.3. Actually, f_1^P may not be constant. So, if f has at least two components then there exist at least two *nonconstant* components. This also holds for $f_{N_p}^P$ in the following Definition 2.4. The energy of such a map is defined by

$$E_+(f) = \sum_i \int\int_{\mathbb{R}\times S^1} |\partial_s f_i^P|^2_{g_J} dsdt + \sum_{i,j} \int_{B_{i,j}} (f_{i,j}^B)^*\omega. \tag{2.5}$$

Definition 2.4. Given a $[x, v] \in \tilde{\mathcal{P}}(H)$ and a $(\Sigma, \underline{l})$ as before, a continuous map $f : \Sigma \setminus \{z_1, \cdots, z_{N_p}\} \to M$ is called a **stable** $(J, H)_-$**-disk** with cap $[x, v]$ if there exist $[x_i, v_i] \in \tilde{\mathcal{P}}(H)$, $i = 1, \cdots, N_p$, with $[x, v] = [x_1, v_1]$, such that:

(1) On each principal component $P_i (i < N_p)$ with cylindrical coordinate (s, t), $f_i^P = f|_{P_i - \{z_i, z_{i+1}\}}$ satisfies (i) (ii) in Definition 2.2(1), but $f_{N_p}^P$ does (2.3) and $f_{N_p}^P(-\infty) = x_{N_p}$ and $E_-(f_{N_p}^P) < +\infty$.

(2) If $N_p > 1$, for each $1 \le i \le N_p - 1$, as relative homology class of (M, x_{i+1}), $[v_i] = [v_{i+1}] + [f_i^P] + \sum_j [f_{i,j}^B]$ and $[v_{N_p}] = [f_{N_p}^P] + \sum_j [f_{N_p,j}^B]$.

(3) All assertions for the bubble components $f_{i,j}^B$ in Definition 2.1(2)(4) still hold. Moreover, all homotopically trivial principal components f_i^P $(i < N_p)$ are not free and do not appear in the next way.

We still define the energy $E_-(f)$ of such a map by the right side of (2.5). As before we may define their equivalence classes. Let us denote by $\langle f, \Sigma, \underline{l}\rangle$ or simply $\langle f\rangle$ the equivalence class of $(f, \Sigma, \underline{l})$. The energy of $\langle f\rangle$ is defined by that of any representative of it. The direct computation shows that

$$E_\pm(f_\pm) \le \mp\mathcal{F}_H([x, v]) + \max|H|. \tag{2.6}$$

The notions of the (effective) dual graph for the stable $(J, H)_\pm$-disks may also be defined with the same way as in [LiuT1]. Let $\overline{\mathcal{M}}_\pm(\tilde{x}; H, J)$ be the spaces of the equivalence classes of all $(J, H)_\pm$-stable disks with cap $\tilde{x}$ respectively. By (2.6),

$$\pm\int_{D^2} v^*\omega \ge -2\max|H| \ \text{ as } \ \overline{\mathcal{M}}_\pm([x, v]; H, J) \ne \emptyset. \tag{2.7}$$

One may equip the weak C^∞ topology on $\overline{\mathcal{M}}_\pm(\tilde{x}; H, J)$ according to the definition given by (i)(ii)(iii) above Proposition 4.1 of [LiuT1] unless we allow the compact set K in (ii) to be able to contain the double point z_- (resp. z_+) on the chain of principal components of the domain of $\langle u_\infty\rangle \in \overline{\mathcal{M}}_+(\tilde{x}; H, J)$ (resp. $\overline{\mathcal{M}}_-(\tilde{x}; H, J)$). Carefully checking proof of Proposition 4.1 in [LiuT1] we have:

Proposition 2.5. *The spaces $\overline{\mathcal{M}}_\pm(\tilde{x};H,J)\supseteq\mathcal{M}_\pm(\tilde{x};H,J)$ are compact and Hausdorff with respect to the weak C^∞-topology.*

Notice that we have two natural continuous maps

$$\mathrm{EV}_\pm(\tilde{x}):\overline{\mathcal{M}}_\pm(\tilde{x};H,J)\to M,\ \langle f_\pm\rangle\mapsto f_\pm(z_\mp) \tag{2.8}$$

with respect to the weak C^∞-topology. Correspondingly, the notions of the dual graphs of the stable $(J,H)_\pm$-disks with a cap $\tilde{x}$ may be introduced in the similar ways. Let $\mathcal{D}_\pm(\tilde{x})$ be the sets of their dual graphs respectively. Both are finite.

Now we are in the position to introduce the notions of positive and negative L^p_k-stable disks with cap $\tilde{x}$. As usual it is always assumed that $k-\frac{2}{p}\ge 1$. In principle, we may proceed as in [LiuT1]. For example, for each $D^+\in\mathcal{D}_+(\tilde{x})$ let $(f,\Sigma,\underline{l})$ be a stable $(J,H)_+$-disk as in Definition 2.2 and with the dual graph D^+. Then a positive L^p_k-stable disk with cap $\tilde{x}$ and of D^+ is a tuple $(\bar{f},\Sigma,\underline{l})$, where $\bar{f}:\Sigma\setminus\{z_2,\cdots,z_{N_p+1}\}\to M$ is locally L^p_k map such that (3) (4) in Definition 2.2 and the following are satisfied:

$(\mathbf{1})'$ $\bar{f}^P_i=\bar{f}|_{P_i-\{z_i,z_{i+1}\}}(i>1)$ satisfy (i) in Definition 2.2(1) and suitable exponential decay condition along ends z_i and z_{i+1} as in [LiuT1], but $\bar{f}^P_1$ only satisfies $\lim_{s\to+\infty}\bar{f}^P_1(s,t)=x_2(t)$ and the exponential decay condition along the end z_2.

We still define its energy $E_+(f)$ by (2.5). The equivalence class of it can also be defined similarly. Denote by $\mathcal{B}^{p,k}_\pm(\tilde{x};H)$ the sets of equivalence classes of those L^p_k-stable disks with cap $\tilde{x}$ and of the dual graphs in $\mathcal{D}_\pm(\tilde{x})$, and whose energy are less than $\mp\mathcal{F}_H(\tilde{x})+\max|H|+1$ (because of (2.6)). As in [LiuT1] one can equip the strong L^p_k-topology on small neighborhoods $\mathcal{W}_\pm(\tilde{x};H,J)$ of $\overline{\mathcal{M}}_\pm(\tilde{x};H,J)$ in $\mathcal{B}^{p,k}_\pm(\tilde{x},H)$ and prove that it is equivalent to the above weak C^∞-topology on $\overline{\mathcal{M}}_\pm(\tilde{x};H,J)$. Once these are well defined we may use Liu-Tian's method to construct the virtual moduli cycles

$$\overline{\mathcal{M}}^{\nu^+}_+(\tilde{x};H,J)=\sum_{I\in\mathcal{N}(\tilde{x})_+}\frac{1}{|\Gamma_I|}\{\pi^+_I:\mathcal{M}^{\nu^+_I}_+(\tilde{x};H,J)\to\mathcal{W}_+(\tilde{x};H,J)\} \tag{2.9}$$

of dimension $\dim M-\mu(\tilde{x})$ in $\mathcal{W}_+(\tilde{x};H,J)$, and

$$\overline{\mathcal{M}}^{\nu_-}_-(\tilde{x};H,J)=\sum_{I\in\mathcal{N}(\tilde{x})_-}\frac{1}{|\Gamma_I|}\{\pi^-_I:\mathcal{M}^{\nu^-_I}_-(\tilde{x};H,J)\to\mathcal{W}_-(\tilde{x};H,J)\} \tag{2.10}$$

of dimension $\mu(\tilde{x})$ in $\mathcal{W}_-(\tilde{x};H,J)$ (cf. [LiuT1, LiuT2] and [LiuT3]). It should also be pointed out that the maps $\mathrm{EV}_\pm$ in (2.8) can naturally extend onto the spaces $\mathcal{W}_\pm(\tilde{x};H,J)$. By composing them with the obvious finite-to-one maps from $\overline{\mathcal{M}}^\nu_\pm(\tilde{x};H,J)$ to $\mathcal{W}_\pm(\tilde{x};H,J)$ that forget the parameterization we get two continuous and stratawise smooth evaluations

$$\mathrm{EV}^{\nu^\pm}_\pm(\tilde{x}):\overline{\mathcal{M}}^{\nu^\pm}_\pm(\tilde{x};H,J)\to M. \tag{2.11}$$

Notice that the choices of different small $\nu^{\pm}$ give the cobordant virtual moduli cycles $\overline{\mathcal{M}}^{\nu^{\pm}}_{\pm}(\tilde{x}; H, J)$ and corresponding evaluations $\mathrm{EV}^{\nu^{\pm}}_{\pm}$.

However, notice that the boundary operator ∂^F in (1.2) depends on the choice of ν. As pointed out below Remark 4.2 of [LiuT1], for ∂^F in (1.2) being indeed a boundary operator the choices of ν in all relative virtual moduli cycles $\widetilde{\mathcal{M}}^{\nu}(\tilde{x}, \tilde{y})$ ($\mu(\tilde{x}) - \mu(\tilde{y}) \le 2$) must satisfy suitable compatible conditions, i.e.,

$$\partial(C(\overline{\mathcal{M}}^{\nu(\tilde{x}\tilde{y})}(\tilde{x}, \tilde{y})) = \sum_{\mu(\tilde{z})=\mu(\tilde{x})-1} C(\overline{\mathcal{M}}^{\nu(\tilde{x}\tilde{z})}(\tilde{x}, \tilde{z})) \times C(\overline{\mathcal{M}}^{\nu(\tilde{z}\tilde{y})}(\tilde{z}, \tilde{y})) \tag{2.12}$$

holds for all pairs $(\tilde{x}, \tilde{y})$ with $\mu(\tilde{y}) - \mu(\tilde{x}) = 2$. Now in order to guarantee that the maps Φ and Ψ constructed in (1.7) and (1.8) commute with ∂^F and the boundary operator ∂^Q in (1.6), we also need carefully choose $\nu^{\pm}$ in (2.9) and (2.10) so that they are compatible with all ν chosen in the definition of ∂^F in (1.2) .

2.2. **Compatible virtual moduli cycles.** In this subsection we shall first complete Liu-Tian's arguments in detail,i.e., proving (2.12), and then outline how to construct all virtual moduli cycles $\overline{\mathcal{M}}^{\nu^{\pm}}_{\pm}(\tilde{x}; H, J)$ compatible with all relative virtual moduli cycles in (1.2). For convenience of the later proof we need to recall briefly the construction of the relative virtual cycle $C(\overline{\mathcal{M}}^{\nu}(\tilde{x}, \tilde{y}))$ in [LiuT1].

Step 1: **Local construction.** For a representative f of $\langle f\rangle \in \overline{\mathcal{M}}(\tilde{x}, \tilde{y}; H, J)$ one may construct a stratified Banach orbifold chart $(\widetilde{W}(f), \Gamma_f, \pi_f)$ around $\langle f\rangle$ in $\mathcal{B}(\tilde{x}, \tilde{y}) = \mathcal{B}^p_k(\tilde{x}, \tilde{y})$, where $\Gamma_f = \mathrm{Aut}(f)$. There exists a natural stratified Banach bundle $\widetilde{\mathcal{L}}(f) \to \widetilde{W}(f)$ with a stratawise smooth right Γ_f-action such that the usual $\bar{\partial}_{J,H}$-operator gives rise to a Γ_f-equivariant stratawise smooth section of this bundle, still denoted by $\bar{\partial}_{J,H}$. Let I denote the dual graph of f and $R(f) \subset (\widetilde{\mathcal{L}}(f)^I)_f$ be the cokernal $\mathrm{coker}(D\bar{\partial}_{J,H}(f))$. Take a smooth cut-off function $\beta_\epsilon(f)$ supported outside of the ϵ-neighborhood of double points of the domain Σ_f. Then each $\nu \in R_\epsilon(f) := \{\beta_\epsilon(f) \cdot \xi \,|\, \xi \in R(f)\}$ may naturally determine a section of the bundle $\widetilde{\mathcal{L}}(f) \to \widetilde{W}(f)$, denoted by $\tilde{\nu}$, such that for each $g \in \widetilde{W}(f)$ the support of $\tilde{\nu}(g) \in (\widetilde{\mathcal{L}}(f))_g$ is away from the gluing region of the domain Σ_g of g.

Step 2: **Global construction.** By the compactness of $\overline{\mathcal{M}}(\tilde{x}, \tilde{y}; H, J)$ one can choose finite points $\langle f_1\rangle, \cdots, \langle f_m\rangle$ such that the union $W := \cup_{i=1}^m W(\langle f_i\rangle)$ is an open neighborhood of $\overline{\mathcal{M}}(\tilde{x}, \tilde{y}; H, J)$ and $\mathcal{L} := \cup_{i=1}^m \mathcal{L}(\langle f_i\rangle)$ is an orbifold bundle over W. Let $\mathcal{N}(\tilde{x}, \tilde{y})$ be the set of all subsets $I = \{i_1, \cdots, i_l\}$ of $\{1, \cdots, m\}$ with $W_I := \cap_{i\in I} W(\langle f_i\rangle) \ne \emptyset$. Let $\pi_i : \widetilde{W}(f_i) \to W(\langle f_i\rangle)$ be the natural projections. For each $I \in \mathcal{N}(\tilde{x}, \tilde{y})$ they defined the group $\Gamma_I := \prod_{i\in I} \Gamma_{f_i}$ and the fiber product

$$\widetilde{W}^{\Gamma_I}_I = \Big\{(u_i)_{i\in I} \in \prod_{i\in I} \widetilde{W}(f_i) \;\Big|\; \pi_i(u_i) = \pi_j(u_j)\ \forall i, j \in I\Big\}. \tag{2.13}$$

Then the projection $\pi_I : \widetilde{W}^{\Gamma_I}_I \to W_I$ has covering group Γ_I. Moreover, for $J \subset I \in \mathcal{N}(\tilde{x}, \tilde{y})$ there is an obvious projection $\pi^I_J : \widetilde{W}^{\Gamma_I}_I \to \widetilde{W}^{\Gamma_J}_J$ satisfying the

relation $\pi_J \circ \pi^I_J = \pi_I$. Repeating the same construction from $\widetilde{\mathcal{L}}(f_i)$ one obtains the bundles $\widetilde{\mathcal{L}}^{\Gamma_I}_I$ and thus a system of bundles $(\widetilde{\mathcal{L}}^\Gamma, \widetilde{W}^\Gamma) = \{(\widetilde{\mathcal{L}}^{\Gamma_I}_I, \widetilde{W}^{\Gamma_I}_I), \pi_I, \pi^I_J \,|\, J \subset I \in \mathcal{N}(\tilde{x}, \tilde{y})\}$. Take open sets $W^1_i \subset\subset W(\langle f_i \rangle)$, $i = 1, \cdots, m$ and the pairs of open sets $W^j_i \subset\subset U^j_i$, $i, j = 1, \cdots, m$, such that $\cup^m_{i=1} W^1_i$ still contains $\overline{\mathcal{M}}(\tilde{x}, \tilde{y}; J, H)$ and that

$$W^1_i \subset\subset U^1_i \subset\subset W^2_i \subset\subset U^2_i \cdots \subset\subset W^m_i \subset\subset W(\langle f_i \rangle). \tag{2.14}$$

In [LiuT1], for each $I \in \mathcal{N}(\tilde{x}, \tilde{y})$ with cardinal number $|I| = k$, they defined

$$V_I = \cap_{i \in I} W^k_i \setminus \cup_{|J|>k} Cl(\cap^k_{j \in J} U^k_j) \tag{2.15}$$

and proved that the open covering $\{V_I \,|\, I \in \mathcal{N}(\tilde{x}, \tilde{y})\}$ of $\overline{\mathcal{M}}(\tilde{x}, \tilde{y}; H, J)$ satisfies

$$\begin{array}{l} V_I \subset W_I \, \forall I \in \mathcal{N}(\tilde{x}, \tilde{y}), \quad \text{and} \\ Cl(V_I) \cap Cl(V_J) \neq \emptyset \, \text{only if } I \subset J \text{ or } J \subset I. \end{array} \tag{2.16}$$

Set $\widetilde{V}_I = (\pi_I)^{-1}(V_I)$ and $\widetilde{E}_I = (\pi_I)^{-1}(\mathcal{L}|_{V_I})$, one gets a system of bundles

$$(\widetilde{E}^\Gamma, \widetilde{V}^\Gamma) = \Big\{(\widetilde{E}^{\Gamma_I}_I, \widetilde{V}^{\Gamma_I}_I), \pi_I, \Gamma_I, \pi^I_J \;\Big|\; J \subset I \in \mathcal{N}(\tilde{x}, \tilde{y})\Big\}. \tag{2.17}$$

Taking Γ_{f_i}-invariant stratawise smooth cut-off function $\gamma(f_i)$ on $\widetilde{W}(f_i)$ such that

$$\gamma(f_i) = 1 \;\text{ in }\; \pi^{-1}_i(W^m_i), \tag{2.18}$$

then each $\nu_i \in R_\epsilon(f_i)$ determines a smooth global section $\bar{\nu}_i$ of $(\widetilde{E}^\Gamma, \widetilde{V}^\Gamma)$. Set

$$R^\epsilon_\delta(\{f_i\}) = \{\nu \in \oplus^m_{i=1} R_\epsilon(f_i) \,|\, |\nu| < \delta\} \tag{2.19}$$

for a small $\delta > 0$. The bundle system $(\widetilde{E}^\Gamma \times R^\epsilon_\delta(\{f_i\}), \widetilde{V}^\Gamma \times R^\epsilon_\delta(\{f_i\}))$ has a well-defined global section

$$\bar{\partial}_{J,H} + e : \, (u_I, \nu) \mapsto \bar{\partial}_{J,H} u_I + \sum^m_{i=1} (\bar{\nu}_i)_I(u_I). \tag{2.20}$$

for any $(u_I, \nu) = (u_I, (\nu_1, \cdots, \nu_m)) \in \widetilde{V}_I \times R^\epsilon_\delta(\{f_i\})$. Moreover, each $\nu \in R^\epsilon_\delta(\{f_i\})$ yields a smooth section $\bar{\partial}^\nu_{J,H} = \{(\bar{\partial}^\nu_{J,H})_I \,|\, I \in \mathcal{N}(\tilde{x}, \tilde{y})\}$ of $(\widetilde{E}^\Gamma, \widetilde{V}^\Gamma)$,

$$(\bar{\partial}^\nu_{J,H})_I(u_I) = \bar{\partial}_{J,H} u_I + \sum^m_{i=1} (\bar{\nu}_i)_I(u_I) \quad \forall u_I \in \widetilde{V}_I. \tag{2.21}$$

Theorem 2.6([LiuT1]). *The section in (2.20) is smooth and transversal to the zero section. Therefore when $\delta > 0$ is small enough, for a generic choice of $\nu \in R^\epsilon_\delta(\{f_i\})$ the section $\bar{\partial}^\nu_{J,H}$ is transversal to the zero section. Thus the family of perturbed moduli spaces $\widetilde{\mathcal{M}}^\nu = \widetilde{\mathcal{M}}^\nu(\tilde{x}, \tilde{y}) = \{\widetilde{\mathcal{M}}^\nu_I = (\bar{\partial}^\nu_{J,H})^{-1}_I(0) \,|\, I \in \mathcal{N}(\tilde{x}, \tilde{y})\}$ is compatible in the sense that $\pi^I_J(\widetilde{\mathcal{M}}^\nu_I) = \widetilde{\mathcal{M}}^\nu_J \cap (\mathrm{Im}\pi^I_J)$ for all $J \subset I \in \mathcal{N}(\tilde{x}, \tilde{y})$.*

Let W^m_i be given by (2.14). For sufficiently small ν we can also require that

$$\widetilde{\mathcal{M}}^\nu_I \subset \cap_{i \in I} \pi^{-1}_i(W^m_i). \tag{2.22}$$

Let $\widetilde{\mathcal{M}}_I^{\nu,D_T}$ (resp. $\widetilde{\mathcal{M}}_I^{\nu,D_B}$) be the top strata (resp. the strata of "broken" connecting orbits) of $\widetilde{\mathcal{M}}_I^{\nu}$. Then $\widetilde{\mathcal{M}}_I^{\nu,c} := \widetilde{\mathcal{M}}_I^{\nu,D_T} \cup \widetilde{\mathcal{M}}_I^{\nu,D_B}$ is a smooth manifold of dimension $\mu(\tilde{x}) - \mu(\tilde{y}) - 2$ and with boundary $\partial\widetilde{\mathcal{M}}_I^{\nu,c} = \widetilde{\mathcal{M}}_I^{\nu,D_B}$. Formally writing

$$C(\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y})) = \sum_{I\in\mathcal{N}(\tilde{x},\tilde{y})}\frac{1}{|\Gamma_I|}\widetilde{\mathcal{M}}_I^{\nu,c}, \qquad \overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y}) = \sum_{I\in\mathcal{N}(\tilde{x},\tilde{y})}\frac{1}{|\Gamma_I|}\widetilde{\mathcal{M}}_I^{\nu},$$

the former was called the relative moduli cycle in [LiuT1]. Later we call the compatible family $\widetilde{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y})$ in Theorem 2.6 a *derived family* for $C(\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y}))$ and $\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y})$. It is clear that $C(\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y})) = \overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y})$ in the case $\mu(\tilde{x}) - \mu(\tilde{y}) \le 2$. For the sake of clearness $C(\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y}))$ will be denoted by $C\big(\overline{\mathcal{M}}^{\nu(\tilde{x}\tilde{y})}(\tilde{x},\tilde{y})\big)$ below. If $\mu(\tilde{x}) - \mu(\tilde{y}) = 2$, then for every $\tilde{z} \in \tilde{\mathcal{P}}(H)$ with $\mu(\tilde{x}) - \mu(\tilde{z}) = 1$ one has also the associated relative moduli cycles $C(\overline{\mathcal{M}}^{\nu(\tilde{x}\tilde{z})}(\tilde{x},\tilde{z}))$ and $C(\overline{\mathcal{M}}^{\nu(\tilde{z}\tilde{y})}(\tilde{z},\tilde{y}))$. The relative virtual moduli cycles satisfying (2.12) are called *compatible*.

Proof of (2.12). To construct such relative virtual cycles, note that there are only finitely many $\tilde{z} \in \tilde{\mathcal{P}}(H)$, saying $\tilde{z}_1,\cdots,\tilde{z}_r$, such that $\mu(\tilde{z}_q) = \mu(\tilde{x}) - 1$ and $\overline{\mathcal{M}}(\tilde{x},\tilde{z}_q;H,J) \ne \emptyset$, $\overline{\mathcal{M}}(\tilde{z}_q,\tilde{y};H,J) \ne \emptyset$ for $q = 1,\cdots,r$. Let $\langle f_s^{(1q)}\rangle \in \overline{\mathcal{M}}(\tilde{x},\tilde{z}_q;H,J)$ and $\langle f_t^{(2q)}\rangle \in \overline{\mathcal{M}}(\tilde{z}_q,\tilde{y};H,J)$, $s = 1,\cdots,m_q$, $t = 1,\cdots,n_q$, be finite points from which one may construct the relative virtual moduli cycles

$$C(\overline{\mathcal{M}}^{\nu(\tilde{x}\tilde{z}_q)}(\tilde{x},\tilde{z}_q)) = \sum_{I\in\mathcal{N}(\tilde{x},\tilde{z}_q)}\frac{1}{|\Gamma_I^{(1q)}|}\widetilde{\mathcal{M}}_I^{\nu(\tilde{x}\tilde{z}_q),c}, \tag{2.23}$$

$$C(\overline{\mathcal{M}}^{\nu(\tilde{z}_q\tilde{y})}(\tilde{z}_q,\tilde{y})) = \sum_{I\in\mathcal{N}(\tilde{z}_q,\tilde{y})}\frac{1}{|\Gamma_I^{(2q)}|}\widetilde{\mathcal{M}}_I^{\nu(\tilde{z}_q\tilde{y}),c}. \tag{2.24}$$

For the future convenience we assume that for $q = 1,\cdots,r$,

$$\begin{aligned} &W^1(\langle f_s^{(1q)}\rangle) \subset\subset U^1(\langle f_s^{(1q)}\rangle)\cdots \subset\subset U^{m_q-1}(\langle f_s^{(1q)}\rangle) \\ &\subset\subset W^{m_q}(\langle f_s^{(1q)}\rangle) \subset\subset W(\langle f_s^{(1q)}\rangle), \quad s = 1,\cdots,m_q, \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} &W^1(\langle f_t^{(2q)}\rangle) \subset\subset U^1(\langle f_t^{(2q)}\rangle)\cdots \subset\subset U^{n_q-1}(\langle f_t^{(2q)}\rangle) \\ &\subset\subset W^{n_q}(\langle f_t^{(2q)}\rangle) \subset\subset W(\langle f_t^{(2q)}\rangle), \quad t = 1,\cdots,n_q, \end{aligned} \tag{2.26}$$

are respectively the open sets as defined in (2.14) that are used to construct the relative virtual moduli cycles in (2.23) and (2.24) above. By (2.15)-(2.17) we get the corresponding bundle systems $(\widetilde{E}^{\Gamma^{(1q)}}, \widetilde{V}^{\Gamma^{(1q)}})$ and $(\widetilde{E}^{\Gamma^{(2q)}}, \widetilde{V}^{\Gamma^{(2q)}})$, $q = 1,\cdots,r$. As in Theorem 2.6 let

$$\begin{aligned} &\nu(\tilde{x}\tilde{z}_q) = \oplus_{s=1}^{m_q}\nu(\tilde{x}\tilde{z}_q)_s \in R_\delta^\epsilon(\{f_s^{(1q)}\}) \quad \text{and} \\ &\nu(\tilde{z}_q\tilde{y}) = \oplus_{t=1}^{n_q}\nu(\tilde{z}_q\tilde{y})_t \in R_\delta^\epsilon(\{f_t^{(2i)}\}) \end{aligned} \tag{2.27}$$

be such that the smooth section $\bar\partial^{\nu(\tilde x\tilde z_q)}_{J,H}$ of the bundle system $(\widetilde E^{\Gamma^{(1q)}},\widetilde V^{\Gamma^{(1q)}})$ and that $\bar\partial^{\nu(\tilde z_q\tilde y)}_{J,H}$ of $(\widetilde E^{\Gamma^{(2q)}},\widetilde V^{\Gamma^{(2q)}})$ are transversal to the zero section respectively. Here $\nu(\tilde x\tilde z_q)_s\in R_\epsilon(f^{(1q)}_s)$ and $\nu(\tilde z_q\tilde y)_t\in R_\epsilon(f^{(2q)}_t)$, $s=1,\cdots,m_q$ and $t=1,\cdots,n_q$. In fact, as in (2.21) we can also require $\nu(\tilde x\tilde z_q)$ and $\nu(\tilde z_q\tilde y)$ so small that

$$\widetilde{\mathcal M}^{\nu(\tilde x\tilde z_q)}_I\subset\cap_{s\in I}(\pi^{(1q)}_s)^{-1}(W^{m_q}(\langle f^{(1q)}_s\rangle))\quad\forall I\in\mathcal N(\tilde x,\tilde z_q),$$
$$\widetilde{\mathcal M}^{\nu(\tilde z_q\tilde y)}_I\subset\cap_{t\in I}(\pi^{(2q)}_t)^{-1}(W^{n_q}(\langle f^{(2q)}_t\rangle))\quad\forall I\in\mathcal N(\tilde z_q,\tilde y).$$

For $q=1,\cdots,r$ we set $f^{(q)}_{st}:=f^{(1q)}_s\sharp_{\tilde z_q}f^{(2q)}_t$, $s=1,\cdots,m_q$, $t=1,\cdots,n_q$. Clearly, these $\langle f^{(q)}_{st}\rangle$ belong to $\overline{\mathcal M}(\tilde x,\tilde y;H,J)$. Moreover, the automorphism group $\Gamma^{(q)}_{st}$ of $f^{(q)}_{st}$ may be identified with the product $\Gamma^{(1q)}_s\times\Gamma^{(2q)}_t$ of the automorphism group $\Gamma^{(1q)}_s$ of $f^{(1q)}_s$ and that $\Gamma^{(2q)}_t$ of $f^{(2q)}_t$. By the construction of the local uniformizer in §2 of [LiuT1] we easily construct a uniformizer $\widetilde W(f^{(q)}_{st})$ such that

$$\widetilde W(f^{(1q)}_s)\sharp\widetilde W(f^{(2q)}_t)\subset\widetilde W(f^{(q)}_{st})\tag{2.28}$$

and that the restriction of $\Gamma^{(q)}_{st}$-action over $\widetilde W(f^{(q)}_{st})$ to $\widetilde W(f^{(1q)}_s)\sharp\widetilde W(f^{(2q)}_t)$ is exactly that of $\Gamma^{(1q)}_s\times\Gamma^{(2q)}_t$ over $\widetilde W(f^{(1q)}_s)\sharp\widetilde W(f^{(2q)}_t)$ in the obvious way, where $\widetilde W(f^{(1q)}_s)\sharp\widetilde W(f^{(2q)}_t)$ denotes the set of all join functions at $\tilde z_q$ of functions in $\widetilde W(f^{(1q)}_s)$ and $\widetilde W(f^{(2q)}_t)$. (One may increase m_q, n_q and shrink $\widetilde W(f^{(1q)}_s)$ and $\widetilde W(f^{(2q)}_t)$ if necessary.) Since each $\overline{\mathcal M}(\tilde x,\tilde z_q;J,H)\sharp\overline{\mathcal M}(\tilde z_q,\tilde y;J,H)$ is compact and $\cup^r_{q=1}\overline{\mathcal M}(\tilde x,\tilde z_q;J,H)\sharp\overline{\mathcal M}(\tilde z_q,\tilde y;J,H)$ are disjoint unions we can require that the above uniformizers $\widetilde W(f^{(q)}_{st})$ satisfy

$$\widetilde W(f^{(q)}_{st})\cap\widetilde W(f^{(q')}_{s't'})=\emptyset\quad\forall q\ne q'.\tag{2.29}$$

By the definition of the index set $\mathcal N(\tilde x,\tilde y)$ above (2.13) it easily follows from (2.29) that the corresponding index set $\mathcal N^0(\tilde x,\tilde y)$ with the collection $\{\widetilde W(f^{(q)}_{st})\,|\,1\le s\le m_q,\ 1\le t\le n_q,\ 1\le q\le r\}$ must have the following form

$$\mathcal N^0(\tilde x,\tilde y)=\cup^r_{q=1}\mathcal N(\tilde x,\tilde z_q)\times\mathcal N(\tilde z_q,\tilde y).\tag{2.30}$$

Notice that every $W(\langle f^{(q)}_{st}\rangle)$ determines an open neighborhood $W(\langle f^{(q)}_{st}\rangle)_1$ of $\langle f^{(1q)}_s\rangle$ in $\mathcal B(\tilde x,\tilde z_q)$ by

$$\left\{\langle g_1\rangle\in\mathcal B(\tilde x,\tilde z_q)\;\middle|\;\exists\langle g_2\rangle\in\mathcal B(\tilde z_q,\tilde y)\ s.t.\ \langle g_1\sharp g_2\rangle\in\mathcal B(\tilde x,\tilde y)\cap W(f^{(q)}_{st})\right\}$$

and that $W(\langle f^{(q)}_{st}\rangle)_2$ of $\langle f^{(2q)}_t\rangle$ in $\mathcal B(\tilde z_q,\tilde y)$ by

$$\left\{\langle g_2\rangle\in\mathcal B(\tilde z_q,\tilde x)\;\middle|\;\exists\langle g_1\rangle\in\mathcal B(\tilde x,\tilde z_q)\ s.t.\ \langle g_1\sharp g_2\rangle\in\mathcal B(\tilde x,\tilde y)\cap W(f^{(q)}_{st})\right\}.$$

Thus (2.28) implies that $W(\langle f^{(1q)}_s\rangle)\subset W(\langle f^{(q)}_{st}\rangle)_1$ and $W(\langle f^{(2q)}_t\rangle)\subset W(\langle f^{(q)}_{st}\rangle)_2$ for all q,s,t. By this and (2.25)(2.26) we can, as in (2.14), choose pairs of open

sets $W^j(\langle f_{st}^{(q)}\rangle)\subset\subset U^j(\langle f_{st}^{(q)}\rangle)$, $j=1,\cdots,m=\sum_{q=1}^r m_q n_q$, such that

$$\begin{aligned}W^1(\langle f_{st}^{(q)}\rangle)\subset\subset U^1(\langle f_{st}^{(q)}\rangle)\subset\subset W^2(\langle f_{st}^{(q)}\rangle)\cdots\\ \subset\subset W^m(\langle f_{st}^{(q)}\rangle)\subset\subset W(\langle f_{st}^{(q)}\rangle)\end{aligned}\tag{2.31}$$

and that $W^{m_q}(\langle f_s^{(1q)}\rangle)\subset W^1(\langle f_{st}^{(q)}\rangle)_1$ and $W^{n_q}(\langle f_t^{(2q)}\rangle)\subset W^1(\langle f_{st}^{(q)}\rangle)_2$ for all q,s,t. As in (2.15) and (2.17) we use (2.30) and (2.31) to construct a bundle system

$$(\widetilde{E},\widetilde{V})=\{(\widetilde{E}_{I_1\times I_2},\widetilde{V}_{I_1\times I_2})\mid I_1\times I_2\in\mathcal{N}^0(\tilde{x},\tilde{y})\}.$$

As in (2.18) we take $\Gamma_{st}^{(q)}$-invariant smooth cut-off function $\gamma(f_{st}^{(q)})$ on $\widetilde{W}(\langle f_{st}^{(q)}\rangle)$ such that $\gamma(f_{st}^{(q)})=1$ in $(\pi_{st}^q)^{-1}(W^m(\langle f_{st}^{(q)}\rangle))$. Note that the supports of $\nu(\tilde{x}\tilde{z}_q)_s$ and $\nu(\tilde{z}_q\tilde{y})_t$ in (2.27) are away from double points on their domains. So the support of $\nu(\tilde{x}\tilde{y})_{st}^q:=\nu(\tilde{x}\tilde{z}_q)_s\sharp\nu(\tilde{z}_q\tilde{y})_t$ is away from all double points of the domain of $f_{st}^{(q)}$ and thus $\nu(\tilde{x}\tilde{y})_{st}^q$ sits in the fibre of $\widetilde{\mathcal{L}}(f_{st}^{(q)})$ at $f_{st}^{(q)}$, $L_{k-1}^p(\wedge^{0,1}((f_{st}^{(q)})^*TM))$. As before we can use $\gamma(f_{st}^{(q)})$ and $\nu(\tilde{x}\tilde{y})_{st}^q$ to get a global section

$$\overline{\nu(\tilde{x}\tilde{y})_{st}^q}=\{(\overline{\nu(\tilde{x}\tilde{y})_{st}^q})_{I_1\times I_2}\mid I_1\times I_2\in\mathcal{N}^0(\tilde{x},\tilde{y})\}$$

of the bundle system $(\widetilde{E},\widetilde{V})$. Let us set

$$\nu(\tilde{x}\tilde{y})^0:=\sum_{q=1}^r\sum_{s=1}^{m_q}\sum_{t=1}^{n_q}\nu(\tilde{x}\tilde{y})_{st}^q\quad\text{and}\quad\overline{\nu(\tilde{x}\tilde{y})^0}:=\sum_{q=1}^r\sum_{s=1}^{m_q}\sum_{t=1}^{n_q}\overline{\nu(\tilde{x}\tilde{y})_{st}^q}.$$

As in (2.21) we get a global section of the bundle system $(\widetilde{E},\widetilde{V})$,

$$\bar{\partial}_{J,H}^{\nu(\tilde{x}\tilde{y})^0}=\left\{\left(\bar{\partial}_{J,H}^{\nu(\tilde{x}\tilde{y})^0}\right)_{I_1\times I_2}=\bar{\partial}_{J,H}+(\overline{\nu(\tilde{x}\tilde{y})^0})_{I_1\times I_2}\;\middle|\;I_1\times I_2\in\mathcal{N}^0(\tilde{x},\tilde{y})\right\}.$$

Then it easily follows from (2.29) and the choice of this section that

$$\widetilde{\mathcal{M}}_{I_1}^{\nu(\tilde{x}\tilde{z}_q)}\sharp\widetilde{\mathcal{M}}_{I_2}^{\nu(\tilde{z}_q\tilde{y})}\subset\widetilde{\mathcal{M}}_{I_1\times I_2}^{\nu(\tilde{x}\tilde{y})^0}:=\left(\left(\bar{\partial}_{J,H}^{\nu(\tilde{x}\tilde{y})^0}\right)_{I_1\times I_2}\right)^{-1}(0)$$

for $I_1\times I_2\in\mathcal{N}(\tilde{x},\tilde{z}_q)\times\mathcal{N}(\tilde{z}_q,\tilde{y})$, and that each section $(\bar{\partial}_{J,H}^{\nu(\tilde{x}\tilde{y})^0})_{I_1\times I_2}$ is transversal to the zero section at all points of

$$\widetilde{\mathcal{M}}_{I_1}^{\nu(\tilde{x}\tilde{z}_q)}\sharp\widetilde{\mathcal{M}}_{I_2}^{\nu(\tilde{z}_q\tilde{y})}\subset\widetilde{W}_{I_1}^{\nu(\tilde{x}\tilde{z}_q)}\sharp\widetilde{W}_{I_2}^{\nu(\tilde{z}_q\tilde{y})}.$$

The last set consists of all points $(u_s\sharp v_t)_{(s,t)\in I_1\times I_2}$ in

$$\prod_{(s,t)\in I_1\times I_2}\widetilde{W}(f_s^{(1q)})\sharp\widetilde{W}(f_t^{(2q)})$$

such that $\pi_{st}^{(q)}(u_s\sharp v_t)=\pi_{s't'}^{(q)}(u_{s'}\sharp v_{t'})$ for any (s,t) and (s',t') in $I_1\times I_2$. By the stability of a surjective map under small perturbation we can show that there exists a $\Gamma_{I_1\times I_2}^{(q)}$-invariant open neighborhoods of $\widetilde{\mathcal{M}}_{I_1}^{\nu(\tilde{x}\tilde{z}_q)}\sharp\widetilde{\mathcal{M}}_{I_2}^{\nu(\tilde{z}_q\tilde{y})}$ in $\widetilde{V}_{I_1\times I_2}$,

$$\widetilde{V}_{I_1\times I_2}^0\subset\subset\widetilde{V}_{I_1\times I_2}^1\tag{2.32}$$

such that the section $(\bar\partial^{\nu(\tilde x\tilde y)^0}_{J,H})_{I_1\times I_2}$ is transversal to the zero section at points of

$$\widetilde V^1_{I_1\times I_2}\cap\widetilde{\mathcal M}^{\nu(\tilde x\tilde y)^0}_{I_1\times I_2}. \tag{2.33}$$

So the space in (2.33) is a cornered and stratified Banach variety that has the dimension given by the Index Theorem on all of its strata. Moreover, the collection

$$\left\{\widetilde{\mathcal M}^{\nu(\tilde x\tilde y)^0}_{I_1\times I_2}\mid I_1\times I_2\in\mathcal N(\tilde x,\tilde z_q)\times\mathcal N(\tilde z_q,\tilde y)\right\}$$

also satisfy the compatibility as in Theorem 2.6. In particular the open neighborhoods in (2.32) may be chosen to guarantee that the collection

$$\left\{\widetilde V^1_{I_1\times I_2}\cap\widetilde{\mathcal M}^{\nu(\tilde x\tilde y)^0}_{I_1\times I_2}\mid I_1\times I_2\in\mathcal N(\tilde x,\tilde z_q)\times\mathcal N(\tilde z_q,\tilde y)\right\}$$

satisfies the compatibility. Note that the projection image of $\widetilde V^1_{I_1\times I_2}\cap\widetilde{\mathcal M}^{\nu(\tilde x\tilde y)^0}_{I_1\times I_2}$ in $\mathcal B(\tilde x,\tilde y)$ is not compact in general (unlike in Theorem 2.6). We denote by

$$\left(\widetilde V^1_{I_1\times I_2}\cap\widetilde{\mathcal M}^{\nu(\tilde x\tilde y)^0}_{I_1\times I_2}\right)^c,$$

the union of the top and 1-codimensional strata of $\widetilde V^1_{I_1\times I_2}\cap\widetilde{\mathcal M}^{\nu(\tilde x\tilde y)^0}_{I_1\times I_2}$, then it is a smooth manifold with boundary of dimension $\mu(\tilde x)-\mu(\tilde y)-1$ that is contained in the smooth locus of $\widetilde V^1_{I_1\times I_2}\cap\widetilde{\mathcal M}^{\nu(\tilde x\tilde y)^0}_{I_1\times I_2}$ and that has the boundary

$$\partial\left(\widetilde V^1_{I_1\times I_2}\cap\widetilde{\mathcal M}^{\nu(\tilde x\tilde y)^0}_{I_1\times I_2}\right)^c=\widetilde{\mathcal M}^{\nu(\tilde x\tilde z_q),D_T}_{I_1}\sharp\widetilde{\mathcal M}^{\nu(\tilde z_q\tilde y),D_T}_{I_2}$$

if $I_1\times I_2\in\mathcal N(\tilde x,\tilde z_q)\times\mathcal N(\tilde z_q,\tilde y)$. In the following we shall extend

$$\left\{\widetilde V^1_{I_1\times I_2}\cap\widetilde{\mathcal M}^{\nu(\tilde x\tilde y)^0}_{I_1\times I_2}\mid I_1\times I_2\in\mathcal N(\tilde x,\tilde y)^0\right\}$$

into a virtual moduli cycle for $\overline{\mathcal M}(\tilde x,\tilde y;J,H)$ under condition that

$$\left\{\widetilde V^0_{I_1\times I_2}\cap\widetilde{\mathcal M}^{\nu(\tilde x\tilde y)^0}_{I_1\times I_2}\mid I_1\times I_2\in\mathcal N(\tilde x,\tilde y)^0\right\}$$

is not changed. Since both $\overline{\mathcal M}(\tilde x,\tilde y;J,H)$ and

$$\cup_{q=1}^r\overline{\mathcal M}(\tilde x,\tilde z_q;J,H)\sharp\overline{\mathcal M}(\tilde z_q,\tilde y;J,H)$$

are compact we can take points $\langle h_1\rangle,\cdots,\langle h_m\rangle$ in

$$\overline{\mathcal M}(\tilde x,\tilde y;J,H)\setminus\cup_{q=1}^r\overline{\mathcal M}(\tilde x,\tilde z_q;J,H)\sharp\overline{\mathcal M}(\tilde z_q,\tilde y;J,H)$$

and their open neighborhoods $W(\langle h_j\rangle)=\pi_j^{\tilde x\tilde y}(\widetilde W(h_j))$, $j=1,\cdots,n$, in $\mathcal B(\tilde x,\tilde y)$ such that all $W(\langle h_j\rangle)$, $W(\langle f^{(q)}_{st}\rangle)$ form an open covering of $\overline{\mathcal M}(\tilde x,\tilde y;J,H)$ satisfying the conditions for the construction of the virtual moduli cycle above. Let us choose Γ_{h_j}-invariant smooth cut-off functions $\gamma(h_j)$ on $\widetilde W(h_j)$ such that

$$\pi_j(\mathrm{suppt}(\gamma(h_j)))\cap V^0_{I_1\times I_2}=\emptyset\ \ \forall 1\le j\le n, I_1\times I_2\in\mathcal N(\tilde x,\tilde y)^0$$

where $V^0_{I_1\times I_2}$ is the projection image of $\widetilde V^0_{I_1\times I_2}$ in $\mathcal B(\tilde x,\tilde y)$.

Let $(\widetilde{E}(\tilde{x}\tilde{y}), \widetilde{V}(\tilde{x}\tilde{y}))$ and $\mathcal{N}(\tilde{x},\tilde{y})$ be the corresponding bundle system and index set to this covering. Assume that

$$\overline{\overline{\nu(\tilde{x}\tilde{y})^q_{st}}} = \left\{(\overline{\overline{\nu(\tilde{x}\tilde{y})^q_{st}}})_I \,|\, I \in \mathcal{N}(\tilde{x},\tilde{y})\right\}$$

is a global section of this bundle system obtained by the cut-off function $\gamma(f^q_{st})$ and $\nu(\tilde{x}\tilde{y})^q_{st}$ as in Step 2. Of course, for each $\nu_j \in R_\epsilon(h_j)$ we still denote by $\bar{\nu}_j = \{(\bar{\nu}_j)_I \,|\, I \in \mathcal{N}(\tilde{x},\tilde{y})\}$ the global section of $(\widetilde{E}(\tilde{x}\tilde{y}), \widetilde{V}(\tilde{x}\tilde{y}))$ obtained from $\gamma(h_j)$ and ν_j as below (2.18).

For $\delta > 0$ we assume that $Z^\epsilon_\delta(\{h_j\})$ is a δ-neighborhood of zero of $\oplus^n_{j=1} R_\epsilon(h_j)$. As before we have a bundle system $(\widetilde{E}(\tilde{x}\tilde{y}) \times Z^\epsilon_\delta(\{h_j\}), \widetilde{V}(\tilde{x}\tilde{y}) \times Z^\epsilon_\delta(\{h_j\}))$ and a well-defined global section of it

$$\bar{\partial}_{J,H} + \hat{e}: \ (u_I, \nu) \mapsto \bar{\partial}_{J,H} u_I + \sum_{j=1}^n (\bar{\nu}_j)_I(u_I) + \sum_{q=1}^r \sum_{s=1}^{m_q} \sum_{t=1}^{n_q} \left(\overline{\overline{\nu(\tilde{x}\tilde{y})^q_{st}}}\right)_I (u_I)$$

for $(u_I, \nu) = (u_I, (\nu_1, \cdots, \nu_n)) \in \widetilde{V}(\tilde{x}\tilde{y}) \times Z^\epsilon_\delta(\{h_j\})$. Notice that our choices above imply this section to be transversal to the zero section for $\delta > 0$ small enough. It follows that for a generic choice of $\nu \in Z^\epsilon_\delta(\{h_j\})$ the section of $(\widetilde{E}(\tilde{x}\tilde{y}), \widetilde{V}(\tilde{x}\tilde{y}))$ given by

$$u_I \mapsto \bar{\partial}_{J,H} u_I + \sum_{j=1}^n (\bar{\nu}_j)_I(u_I) + \sum_{q=1}^r \sum_{s=1}^{m_q} \sum_{t=1}^{n_q} (\overline{\overline{\nu(\tilde{x}\tilde{y})^q_{st}}})_I (u_I) \tag{2.34}$$

for $u_I \in \widetilde{V}(\tilde{x}\tilde{y})_I$, is transversal to the zero section.

Setting $\nu(\tilde{x}\tilde{y}) := (\oplus^r_{q=1} \oplus^{m_q}_{s=1} \oplus^{n_q}_{t=1} \nu(\tilde{x}\tilde{y})^q_{st}) \oplus (\oplus^n_{j=1} \nu_j)$ then it belongs to $(\oplus^r_{q=1} \oplus^{m_q}_{s=1} \oplus^{n_q}_{t=1} R_\epsilon(f^{(1q)}_s) \sharp R_\epsilon(f^{(2q)}_t)) \oplus (\oplus^n_{j=1} R_\epsilon(h_j))$, and the section

$$\bar{\partial}^{\nu(\tilde{x}\tilde{y})}_{J,H} = \left\{ \left(\bar{\partial}^{\nu(\tilde{x}\tilde{y})}_{J,H}\right)_I \,|\, I \in \mathcal{N}(\tilde{x},\tilde{y})\right\}$$

of $(\widetilde{E}(\tilde{x}\tilde{y}), \widetilde{V}(\tilde{x}\tilde{y}))$ is transversal to the zero section. Here $(\bar{\partial}^{\nu(\tilde{x}\tilde{y})}_{J,H})_I$ is given by (2.34). Denote by $C(\overline{\mathcal{M}}^{\nu(\tilde{x}\tilde{y})}(\tilde{x},\tilde{y}))$ the virtual moduli cycle constructed from this section. It is easy to see that its boundary is given by

$$\sum_{q=1}^r \sum_{I_1 \in \mathcal{N}(\tilde{x},\tilde{z}_q)} \sum_{I_2 \in \mathcal{N}(\tilde{z}_q,\tilde{y})} \frac{1}{|\Gamma^{(1q)}_{I_1}| \cdot |\Gamma^{(2q)}_{I_2}|} \widetilde{\mathcal{M}}^{\nu(\tilde{x}\tilde{z}_q), D_T}_{I_1} \sharp \widetilde{\mathcal{M}}^{\nu(\tilde{z}_q\tilde{y}), D_T}_{I_2},$$

which may be identified with

$$\sum_{q=1}^r \sum_{I_1 \in \mathcal{N}(\tilde{x},\tilde{z}_q)} \sum_{I_2 \in \mathcal{N}(\tilde{z}_q,\tilde{y})} \frac{1}{|\Gamma^{(1q)}_{I_1}| \cdot |\Gamma^{(2q)}_{I_2}|} \widetilde{\mathcal{M}}^{\nu(\tilde{x}\tilde{z}_q), D_T}_{I_1} \times \widetilde{\mathcal{M}}^{\nu(\tilde{z}_q\tilde{y}), D_T}_{I_2}.$$

But the latter is just $\sum^r_{q=1} C(\overline{\mathcal{M}}^{\nu(\tilde{x}\tilde{z}_q)}(\tilde{x},\tilde{z}_q)) \times C(\overline{\mathcal{M}}^{\nu(\tilde{z}_q\tilde{y})}(\tilde{z}_q,\tilde{y}))$. (2.12) is proved. □

Now we first construct all relative virtual moduli cycles $C(\overline{\mathcal{M}}^{\nu(\tilde{x}\tilde{y})}(\tilde{x},\tilde{y}))$ for all $\tilde{x},\tilde{y}\in\tilde{\mathcal{P}}(H)$ with $\mu(\tilde{x})-\mu(\tilde{y})=1$, such that

$$\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y})))=\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{x}\sharp(-A),\tilde{y}\sharp(-A))))\tag{2.35}$$

for any $A\in\Gamma$. Then we follow the above methods to construct all relative virtual moduli cycles $C(\overline{\mathcal{M}}^{\nu(\tilde{x}\tilde{y})}(\tilde{x},\tilde{y}))$ for all $\tilde{x},\tilde{y}\in\tilde{\mathcal{P}}(H)$ with $\mu(\tilde{x})-\mu(\tilde{y})=2$. Such the family of the relative virtual moduli cycles is compatible and thus satisfies the requirements in the definition of ∂^F.

Remark 2.7. As in [LiuT2] and [LiuT3] we need the virtual moduli cycles of dimension more than one in this paper, and can use the notion of local components in [LiuT3] to construct a desingularization of the bundle system $(\widetilde{\mathcal{L}}^{\Gamma},\widetilde{W}^{\Gamma})$, a new bundle system $(\widehat{\mathcal{L}}^{\Gamma},\widehat{W}^{\Gamma})=\{(\widehat{\mathcal{L}}^{\Gamma_I}_I,\widehat{W}^{\Gamma_I}_I)\,|\,I\in\mathcal{N}(\tilde{x},\tilde{y})\}$ such that each $\widehat{\mathcal{L}}^{\Gamma_I}_I$ (resp. $\widehat{W}^{\Gamma_I}_I$) is a stratified Banach manifold (resp. bundle). Then replacing $(\widetilde{\mathcal{L}}^{\Gamma},\widetilde{W}^{\Gamma})$ everywhere by $(\widehat{\mathcal{L}}^{\Gamma},\widehat{W}^{\Gamma})$ in the previous construction of the virtual moduli cycles one can get a virtual moduli cycle

$$\sum_{I\in\mathcal{N}(\tilde{x},\tilde{y})}\frac{1}{|\Gamma_I|}\Big\{\pi_I:\widehat{\mathcal{M}}^{\nu}_I\to W\Big\},\ \text{ still denoted by }\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{y}),$$

such that each $\widehat{\mathcal{M}}^{\nu}_I$ is a cornered smooth manifold. We can use the same method to make suitable modifications for the above arguments and in the case $\mu(\tilde{x})-\mu(\tilde{y})=2$ obtain the conclusion corresponding with (2.12), i.e.,

$$\partial\overline{\mathcal{M}}^{\nu(\tilde{x}\tilde{y})}(\tilde{x},\tilde{y})=\textstyle\sum_{\mu(\tilde{z})=\mu(\tilde{x})-1}\overline{\mathcal{M}}^{\nu(\tilde{x}\tilde{z})}(\tilde{x},\tilde{z})\times\overline{\mathcal{M}}^{\nu(\tilde{z}\tilde{y})}(\tilde{z},\tilde{y}).$$

Hence we can replace $\sharp C(\overline{\mathcal{M}}^{\nu(\tilde{x}\tilde{y})}(\tilde{x},\tilde{y}))$ by $\sharp\overline{\mathcal{M}}^{\nu(\tilde{x}\tilde{y})}(\tilde{x},\tilde{y})$ in (1.2).

Now we begin to construct all virtual moduli cycles $\overline{\mathcal{M}}^{\nu^{\pm}}_{\pm}(\tilde{x};H,J)$ in (2.9) and (2.10) which are compatible with the relative virtual cycles used in (1.2). We only consider $\overline{\mathcal{M}}^{\nu^{+}}_{+}(\tilde{x};H,J)$. Denote by $T\overline{\mathcal{M}}^{\nu^{+}}_{+}(\tilde{x};H,J)$ and $B\overline{\mathcal{M}}^{\nu^{+}}_{+}(\tilde{x};H,J)$ its top strata and 1-codimensional strata. If $\dim\overline{\mathcal{M}}^{\nu^{+}}_{+}(\tilde{x};H,J)=\dim M-\mu(\tilde{x})>0$, then any element f_+ of $B\mathcal{M}^{\nu^{+}_I}_{+}(\tilde{x};H,J)$ must have a form $f_+=(h_+,g)$. Here $g\in\widetilde{\mathcal{M}}^{\nu_2}_{I_2}(\tilde{y},\tilde{x};H,J)$ and $\mu(\tilde{y})-\mu(\tilde{x})=1$, and $h_+\in T\mathcal{M}^{\nu^{+}_{I_1}}_{+}(\tilde{y};H,J)$. So we can construct each $\overline{\mathcal{M}}^{\nu^{+}}_{+}(\tilde{x};H,J)$ inductively with respect to $\dim M-\mu(\tilde{x})$. In fact, if these have been constructed for $\dim M-\mu(\tilde{x})=0$, then for each $\tilde{x}\in\tilde{\mathcal{P}}(H)$ with $\dim M-\mu(\tilde{x})=1$, we can use them and $\widetilde{\mathcal{M}}^{\nu}(\tilde{y},\tilde{x};H,J)$ used in (1.2) to construct

$$B\mathcal{M}^{\nu^{+}}_{+}(\tilde{x};H,J):=\cup_{\mu(\tilde{y})=\mu(\tilde{x})-1}T\mathcal{M}^{\nu^{+}}_{+}(\tilde{y};H,J)\times\widetilde{\mathcal{M}}^{\nu}(\tilde{y},\tilde{x};H,J).$$

Next as done in the proof of (2.12) we can extend $B\mathcal{M}^{\nu^{+}}_{+}(\tilde{x};H,J)$ into a derived family $\mathcal{M}^{\nu^{+}}_{+}(\tilde{x};H,J)=\{\mathcal{M}^{\nu^{+}_I}_{+}(\tilde{x};H,J)\,|\,I\in\mathcal{N}^{+}(\tilde{x})\}$ for a virtual moduli cycle

$\overline{\mathcal{M}}^{\nu^+}_+(\tilde{x};H,J)$. Let $\{\overline{\mathcal{M}}^{\nu^+}_+(\tilde{x};H,J)\,|\,\tilde{x}\in\tilde{\mathcal{P}}(H)\}$ be all virtual moduli cycles constructed by induction. Then it holds that

$$B\mathcal{M}^{\nu^+}_+(\tilde{x};H,J)=\cup_{\mu(\tilde{y})=\mu(\tilde{x})-1}T\mathcal{M}^{\nu^+}_+(\tilde{y};H,J)\times\widetilde{\mathcal{M}}^{\nu}(\tilde{y},\tilde{x};H,J).$$

Consider the natural orientations on them again we can write

$$B\overline{\mathcal{M}}^{\nu^+}_+(\tilde{x};H,J)=\sum_{\mu(\tilde{y})=\mu(\tilde{x})+1}\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{y},\tilde{x})))\cdot T\overline{\mathcal{M}}^{\nu^+}_+(\tilde{y};H,J). \tag{2.36}$$

Here the rational numbers $\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{y},\tilde{x})))$ is as in (1.2), and the identity in (2.36) is understood as follows: Since the sums at the right side of (2.36) are finite we can take $L>0$ to be the smallest common multiple of denominators of all rational numbers $\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{y},\tilde{x})))$. Then (2.36) is equivalent to

$$L\cdot B\overline{\mathcal{M}}^{\nu^+}_+(\tilde{x};H,J)=\sum_{\mu(\tilde{y})=\mu(\tilde{x})+1}(L\cdot\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{y},\tilde{x}))))\cdot T\overline{\mathcal{M}}^{\nu^+}_+(\tilde{y};H,J).$$

For this identity the left side is understood as the disjoint union of L copies of $B\overline{\mathcal{M}}^{\nu^+}_+(\tilde{x};H,J)$, and the right side is also the disjoint union that contains, for each $\tilde{y}=[y,u]$ with $\mu(\tilde{y})=\mu(\tilde{x})+1$, $L\cdot\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{y},\tilde{x})))$ copies of $T\overline{\mathcal{M}}^{\nu^+}_+(\tilde{y};H,J)$ as $\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{y},\tilde{x})))>0$, and $(-L)\cdot\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{y},\tilde{x})))$ copies of $T\overline{\mathcal{M}}^{\nu^+}_+(\tilde{y};H,J)^*$ as $\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{y},\tilde{x})))<0$, where $T\overline{\mathcal{M}}^{\nu^+}_+(\tilde{y};H,J)^*$ is $T\overline{\mathcal{M}}^{\nu^+}_+(\tilde{y};H,J)$ with the orientation reversed. In other words, it is understood as the topological sum. With the same methods we can construct the compatible $\overline{\mathcal{M}}^{\nu^-}_-(\tilde{x};H,J)$ and obtain

$$B\overline{\mathcal{M}}^{\nu^-}_-(\tilde{x};H,J)=\sum_{\mu(\tilde{z})=\mu(\tilde{x})-1}\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{z})))\cdot T\overline{\mathcal{M}}^{\nu^-}_-(\tilde{z};H,J). \tag{2.37}$$

3. THE INTERSECTIONS OF VIRTUAL MODULI CYCLES WITH STABLE AND UNSTABLE MANIFOLDS

For the materials on Morse homology the readers may refer to [AuB, Sch1] and [Sch4]. Given a Morse-Smale pair (h,g) on M with $h\in\mathcal{O}(h_0)$ the stable and unstable manifolds of a critical point $a\in\mathrm{Crit}(h)$ are given by

$$W^s(a,h,g):=\{\gamma:[0,\infty)\to M\,|\,\dot\gamma(s)+\nabla^g h(\gamma(s))=0,\ \gamma(\infty)=a\},$$
$$W^u(a,h,g):=\{\gamma:(-\infty,0]\to M\,|\,\dot\gamma(s)+\nabla^g h(\gamma(s))=0,\ \gamma(-\infty)=a\}.$$

There are two obvious evaluations

$$\begin{aligned}E^s_a&:W^s(a,h,g)\to M,\ \gamma\mapsto\gamma(0)\quad\text{and}\\ E^u_a&:W^u(a,h,g)\to M,\ \gamma\mapsto\gamma(0).\end{aligned} \tag{3.1}$$

Both are also smooth embeddings into M. Throughout this paper we fix orientations for all unstable manifolds $W^u(a,h,g)$, then the orientation of M induces orientations for $W^s(a,h,g)$ and $W^u(a,h,g)\cap W^s(b,h,g)$. We wish to study the intersection numbers of the maps in (2.11) and (3.1) under some conditions. As

usual we need their compactifications of the following versions. Consider the disjoint union $\overline{W}^u(a,h,g) := W^u(a,h,g) \cup S\overline{W}^u(a,h,g)$, where $S\overline{W}^u(a,h,g)$ is the disjoint unions

$$\cup \widehat{M}_{a,a_1}(h,g) \times \cdots \times \widehat{M}_{a_{i-1},a_i}(h,g) \times W^u(a_i,h,g)$$

for all critical points $a_0 = a, \cdots, a_i$ with the Morse indexes $\mu(a_0) > \cdots > \mu(a_i)$. Similarly, in the disjoint union $\overline{W}^s(a,h,g) := W^s(a,h,g) \cup S\overline{W}^s(a,h,g)$ the second set $S\overline{W}^s(a,h,g)$ is the disjoint unions

$$\cup W^s(a_i,h,g) \times \widehat{M}_{a_i,a_{i-1}}(h,g) \times \cdots \times \widehat{M}_{a_1,a}(h,g)$$

for critical points $a_0 = a, \cdots, a_i$ with the Morse indexes $\mu(a_0) < \cdots < \mu(a_i)$. As usual the compactness and gluing arguments in Morse homology (see [AuB] and [Sch1]) give:

Lemma 3.1. *The sets $\overline{W}^u(a,h,g)$ and $\overline{W}^s(a,h,g)$ may be topologized with a natural way so that they are the compactifications of $W^u(a,h,g)$ and $W^s(a,h,g)$ respectively, and that $\partial\overline{W}^u(a,h,g) = S\overline{W}^u(a,h,g)$ and $\partial\overline{W}^s(a,h,g) = S\overline{W}^s(a,h,g)$. Moreover these compactified spaces both have the structure of a manifold with corners, and maps E^u_a and E^s_a may smoothly extend to them, denoted by $\bar{E}^u_a$ and $\bar{E}^s_a$, which also give natural injective immersions from these two compactified spaces into M.*

The following lemma, slightly different from Theorem 4.9 in [Sch4], may be easily proved (cf. [Lu2]).

Lemma 3.2. *Let $\mathcal{R}$ be the set of all Riemannian metrics on M. Then for any smooth map χ from a smooth manifold V to M there exists a Baire subset $(\mathcal{O} \times \mathcal{R})_{\rm reg} \subset \mathcal{O}(h_0) \times \mathcal{R}$ such that for each pair $(h,g) \in (\mathcal{O}(h_0) \times \mathcal{R})_{\rm reg}$ and $a \in \mathrm{Crit}(h)$ the maps E^s_a and E^u_a are transverse to χ. Consequently, the spaces*

$$\mathcal{M}^s_{\chi,a}(h,g) := \{(p,\gamma) \in V \times W^s(a,h,g) \,|\, \chi(p) = E^s_a(\gamma) = \gamma(0)\},$$
$$\mathcal{M}^u_{\chi,a}(h,g) := \{(p,\gamma) \in V \times W^u(a,h,g) \,|\, \chi(p) = E^u_a(\gamma) = \gamma(0)\}$$

are respectively smooth manifolds of $\dim V - \mu(a)$ and $\dim V + \mu(a) - 2n$.

As usual, if V, M, $W^s(a,h,g)$ and $W^u(a,h,g)$ are oriented then $\mathcal{M}^s_{\chi;a}(h,g)$ and $\mathcal{M}^u_{\chi;a}(h,g)$ have the natural induced orientations. Specially, if $\mathcal{M}^s_{\chi;a}(h,g)$ and $\mathcal{M}^u_{\chi;a}(h,g)$ are of dimension zero we have the oriented intersection numbers $\chi \cdot E^s_a$ and $\chi \cdot E^u_a$ which counts the algebraic sum of the oriented points in $\mathcal{M}^s_{\chi;a}(h,g)$ and $\mathcal{M}^u_{\chi;a}(h,g)$ respectively. Note that the 1-codimensional stratum of $\overline{W}^u(a,h,g)$ (resp. $\overline{W}^s(a,h,g)$) is given by

$$\sum_{\mu(b)=\mu(a)-1} n(a,b) \cdot W^u(b,h,g) \quad \left(\text{resp.} \sum_{\mu(c)=\mu(a)+1} n(c,a) \cdot W^s(c,h,g)\right),$$

where $n(a,b)$ and $n(c,a)$ are as in (1.6).

Since there are only countable manifolds involved in our arguments using Claim A.1.11 in [LO] we may always fix a $h \in \mathcal{O}(h_0)$ such that all transversal arguments hold for a generic Riemannian metric g on M. By lemmas 3.1 and 3.2 and these remarks we immediately obtain:

Proposition 3.3. *If $\mu(a) = \mu(\tilde{x})$ then for a generic Riemannian metric g on M the maps $\bar{E}^u_a$ and $\mathrm{EV}^{\nu^+}_+(\tilde{x})$, and $\bar{E}^s_a$ and $\mathrm{EV}^{\nu^-}_-(\tilde{x})$ are transversal, and their intersection numbers are the well-defined rational numbers $\bar{E}^u_a \cdot \mathrm{EV}^{\nu^+}_+(\tilde{x})$ and $\bar{E}^s_a \cdot \mathrm{EV}^{\nu^-}_-(\tilde{x})$.*

From now on, for the sake of clearness we denote by

$$\begin{aligned} n^{\nu^+}_+(a,\tilde{x}) &\equiv n^{\nu^+}_+(a,\tilde{x};H,J;h,g) := \bar{E}^u_a \cdot \mathrm{EV}^{\nu^+}_+(\tilde{x}),\\ n^{\nu^-}_-(a,\tilde{x}) &\equiv n^{\nu^-}_-(a,\tilde{x};H,J;h,g) := \bar{E}^s_a \cdot \mathrm{EV}^{\nu^-}_-(\tilde{x}). \end{aligned} \tag{3.2}$$

The standard arguments show that these numbers are independent of small regular $\nu^\pm$. Later, we state no longer this and often omit $\nu^\pm$ without occurrence of confusions. It easily follows from (2.7) that:

Proposition 3.4. *If $n_\pm(a,[x,v]) \neq 0$ then $\pm\int_{D^2} v^*\omega \geq -2\max|H|$.*

Using (2.36), (2.37) and lemmas 3.1 and 3.2 we may obtain the following two results:

Proposition 3.5. *If $\mu(a) - \mu(\tilde{x}) = 1$ then for a generic Riemannian metric g on M the fibre product*

$$\overline{W}^u(a,h,g) \times_{\bar{E}^u_a = \mathrm{EV}^{\nu^+}_+(\tilde{x})} \overline{\mathcal{M}}^{\nu^+}_+(\tilde{x};H,J)$$

is still a collection of compatible local cornered smooth manifolds of dimension 1 and with the natural orientations. Its boundary is given by

$$\begin{aligned} &\big(\cup_{\mu(b)=\mu(a)-1} n(a,b)\cdot\big(W^u(b,h,g) \times_{E^u_b = \mathrm{EV}^{\nu^+}_+(\tilde{x})} \overline{\mathcal{M}}^{\nu^+}_+(\tilde{x};H,J)\big)\big)\cup\\ &\big(-\cup_{\mu(\tilde{y})=\mu(\tilde{x})+1} \sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{y},\tilde{x})))\cdot\big(\overline{W}^u(a,h,g) \times_{\bar{E}^u_a = \mathrm{EV}^{\nu^+}_+(\tilde{y})} \overline{\mathcal{M}}^{\nu^+}_+(\tilde{y};H,J)\big)\big). \end{aligned}$$

Notice that the projection onto $M \times \mathcal{W}_+(\tilde{x};H,J)$ of this set is a finite set. Let us denote by $\sharp(\partial(\overline{W}^u(a,h,g) \times_{\bar{E}^u_a = \mathrm{EV}^{\nu^+}_+(\tilde{x})} \overline{\mathcal{M}}^{\nu^+}_+(\tilde{x};H,J)))$ the number of elements of this finite set counted with appropriate signs and rational weights. Then this number must be zero, and it follow that

$$\sum_{\mu(b)=\mu(a)-1} n(a,b)\cdot n^{\nu^+}_+(b,\tilde{x}) = \sum_{\mu(\tilde{y})=\mu(\tilde{x})+1} n^{\nu^+}_+(a,\tilde{y})\cdot\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{y},\tilde{x}))).$$

Remark that the fibre product in Proposition 3.5 has boundary. But we assume its dimension to be 1. Hence its boundary agrees with the 1-codimensional stratum of the fibre product. This remark is still valid for the following proposition:

Proposition 3.6. *If $\mu(\tilde{x}) - \mu(a) = 1$ then for a generic Riemannian metric g on M the fibre product*

$$\overline{W}^s(a,h,g) \times_{\bar{E}^s_a = \mathrm{EV}^{\nu^-}_-(\tilde{x})} \overline{\mathcal{M}}^{\nu^-}_-(\tilde{x};H,J)$$

is a collection of compatible local cornered smooth manifolds of dimension 1 and with the natural orientations. Its boundary is given by

$$\big(\cup_{\mu(a)=\mu(b)-1} n(b,a)\cdot \big(W^s(b,h,g) \times_{E^u_b = \mathrm{EV}^{\nu^-}_-(\tilde{x})} \overline{\mathcal{M}}^{\nu^-}_-(\tilde{x};H,J)\big)\big)\cup$$
$$\big(-\cup_{\mu(\tilde{z})=\mu(\tilde{x})-1} \sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{z})))\cdot \big(\overline{W}^u(a,h,g) \times_{\bar{E}^u_a = \mathrm{EV}^{\nu^-}_-(\tilde{z})} \overline{\mathcal{M}}^{\nu^-}_-(\tilde{z};H,J)\big)\big).$$

Consequently, $\sharp(\partial(\overline{W}^s(a,h,g) \times_{\bar{E}^u_a = \mathrm{EV}^{\nu^-}_-(\tilde{x})} \overline{\mathcal{M}}^{\nu^-}_-(\tilde{x};H,J))) = 0$ *implies that*

$$\sum_{\mu(\tilde{z})=\mu(\tilde{x})-1} \sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{x},\tilde{z})))\cdot n^{\nu^-}_-(a,\tilde{z}) = \sum_{\mu(b)=\mu(a)+1} n^{\nu^-}_-(b,\tilde{x})\cdot n(b,a).$$

Remark 3.7. Using Proposition 3.4 it is easily proved that Φ and Ψ are Λ_ω-module homomorphisms(see [Lu2]). Moreover, using Propositions 3.4, 3.5, 3.6 and (2.35) we may prove that $\Phi\circ\partial^Q_k = \partial^F_k\circ\Phi$ and $\Psi\circ\partial^F_k = \partial^Q_k\circ\Psi$ for every integer k. That is, Φ and Ψ also induce the homomorphisms between two corresponding homology groups(see [Lu2]).

4. Proof of Theorem 1.1

Theorem 1.1 may follow from the following Theorems 4.1 and 4.9 immediately.

Theorem 4.1. *$\Psi\circ\Phi$ is chain homotopy equivalent to the identity. Consequently, Φ induces an injective Λ_ω-module homomorphism from $QH_*(h,g;\mathbb{Q})$ to $HF_*(M,\omega;H,J,\nu;\mathbb{Q})$.*

Proof. We shall prove it in four steps.
Step 1. For every $\langle a,A\rangle\in \mathrm{Crit}(h)\times\Gamma$ with $\mu(\langle a,A\rangle)=k$ we have

$$\Psi\circ\Phi(\langle a,A\rangle) = \sum_{\mu(\langle b,B\rangle)=k} m^{\nu}_{+,-}(\langle a,A\rangle;\langle b,B\rangle)\langle b,B\rangle,$$

where $m^{\nu}_{+,-}(\langle a,A\rangle;\langle b,B\rangle)$ is given by

$$\sum_{\mu(\tilde{x})=k} n^{\nu^+}_+(a,\tilde{x}\sharp(-A))\cdot n^{\nu^-}_-(b,\tilde{x}\sharp(-B)). \tag{4.1}$$

Notice here that $\langle a,A\rangle$ and $\langle b,B\rangle$ satisfy

$$\mu(a)-\mu(b)+2c_1(B-A)=0. \tag{4.2}$$

Firstly, we claim that the sum in (4.1) is finite. In fact, if $\tilde{x}\in\tilde{\mathcal{P}}_k(H)$ is such that

$$n^{\nu^+}_+(a,\tilde{x}\sharp(-A))\cdot n^{\nu^-}_-(b,\tilde{x}\sharp(-B))\neq 0, \tag{4.3}$$

by Proposition 3.4 we have

$$\omega(A) - 2\max|H| \le \int_{D^2} v^*\omega \le \omega(B) + 2\max|H|. \tag{4.4}$$

It easily follows that the number of such $\tilde{x} \in \tilde{\mathcal{P}}(H)$ is finite. Let them be $\tilde{x}_1, \cdots, \tilde{x}_s$. Then for a generic $(h,g) \in \mathcal{O}(h_0) \times \mathcal{R}$, it holds that

$$m^{\nu}_{+,-}(\langle a, A\rangle; \langle b, B\rangle) = \sum_{i=1}^{s} n_+^{\nu^+}(a, \tilde{x}_i\sharp(-A)) \cdot n_-^{\nu^-}(b, \tilde{x}_i\sharp(-B)). \tag{4.5}$$

Note that for each $\tilde{x} \in \tilde{\mathcal{P}}_k(H)$ the product $n_+^{\nu^+}(a, \tilde{x}\sharp(-A)) \cdot n_-^{\nu^-}(b, \tilde{x}\sharp(-B))$ can be explained as the rational intersection number

$$(\bar{E}^u_a \times \bar{E}^s_b) \cdot (\mathrm{EV}_+^{\nu^+}(\tilde{x}\sharp(-A)) \times \mathrm{EV}_-^{\nu^-}(\tilde{x}\sharp(-B))) \tag{4.6}$$

of the product map

$$\mathrm{EV}_+^{\nu^+}(\tilde{x}\sharp(-A)) \times \mathrm{EV}_-^{\nu^-}(\tilde{x}\sharp(-B))$$

from $\overline{\mathcal{M}}_+^{\nu^+}(\tilde{x}\sharp(-A); H, J) \times \overline{\mathcal{M}}_-^{\nu^-}(\tilde{x}\sharp(-B); H, J)$ to $M \times M$ and the evaluation

$$\bar{E}^u_a \times \bar{E}^s_b : \overline{W}^u(a,h,g) \times \overline{W}^s(b,h,g) \to M \times M \tag{4.7}$$

for a generic $(h,g) \in \mathcal{O}(h_0) \times \mathcal{R}$. But the intersection number in (4.6) is the number of the oriented points with rational weights in the fibre product

$$(\overline{W}^u(a,h,g) \times \overline{W}^s(b,h,g)) \times_R (\overline{\mathcal{M}}_+^{\nu^+}(\tilde{x}\sharp(-A)) \times \overline{\mathcal{M}}_-^{\nu^-}(\tilde{x}\sharp(-B))). \tag{4.8}$$

Here R is representing $\mathrm{EV}_+^{\nu^+} \times \mathrm{EV}_-^{\nu^-} = \bar{E}^u_a \times \bar{E}^s_b$, and we have omitted H, J in $\overline{\mathcal{M}}_\pm^{\nu^\pm}(\cdot\,; H, J)$. However, because of dimension relations the fibre product in (4.8) is an empty set for a generic $(h,g) \in \mathcal{O}(h_0) \times \mathcal{R}$ if $\mu([x,v]) \ne k$. As usual we still understand the intersection number being zero in this case. Hence (4.1) and (4.5) become

$$\begin{aligned} &m^{\nu}_{+,-}(\langle a, A\rangle; \langle b, B\rangle) \\ =&\textstyle\sum_{\tilde{x} \in \tilde{\mathcal{P}}(H)} (\bar{E}^u_a \times \bar{E}^s_b) \cdot (\mathrm{EV}_+^{\nu^+}(\tilde{x}\sharp(-A)) \times \mathrm{EV}_-^{\nu^-}(\tilde{x}\sharp(-B))). \end{aligned} \tag{4.9}$$

Step 2. We need to give another explanation of the intersection number in (4.6). To this goal we note that for a given $D \in \Gamma$ a positive disk $u_+ \in \mathcal{M}_+(\tilde{y}; H, J)$ and a negative disk $u_- \in \mathcal{M}_-(\tilde{y}\sharp(-D); H, J)$ may be glued into a sphere in M along y, denoted by $u_+\sharp u_-$. Using (2.2) and (2.4) one easily checks that it is a representative of D. Let us denote by $\mathcal{M}_D(\tilde{y}; H, J)$ by the space of all such $u_+\sharp u_-$. Clearly, it can be identified with the product space $\mathcal{M}_+(\tilde{y}; H, J) \times \mathcal{M}_-(\tilde{y}\sharp(-D); H, J)$. Therefore, its virtual dimension is equal to $\dim M + 2c_1(D)$. We denote by

$$\mathcal{M}_D(H, J) = \cup_{\tilde{y} \in \tilde{\mathcal{P}}(H)} \mathcal{M}_D(\tilde{y}; H, J).$$

As in (4.3) and (4.4) it is easy to prove that the union at the right side is only finite union. That is, there only exist finitely many $\tilde{y}_1, \cdots, \tilde{y}_r$ in $\tilde{\mathcal{P}}(H)$ such that $\mathcal{M}_D(\tilde{y}_i; H, J) \neq \emptyset$ for $i = 1, \cdots, r$. Thus

$$\mathcal{M}_D(H,J) = \cup_{i=1}^r \mathcal{M}_D(\tilde{y}_i; H, J). \tag{4.10}$$

In particular, (4.3) and (4.4) imply that

$$\mathcal{M}_{B-A}(H,J) = \cup_{i=1}^s \mathcal{M}_{B-A}(\tilde{x}_i \sharp(-A); H, J). \tag{4.11}$$

Note that the right sides of both (4.10) and (4.11) are all disjoint unions. In order to compactify $\mathcal{M}_D(H,J)$ we introduce:

Definition 4.2. Given $\tilde{y} \in \tilde{\mathcal{P}}(H)$, $D \in \Gamma$, a semistable $\mathcal{F}$-curve $(\Sigma, \underline{l})$ with at least two principal components (cf. Def.2.1), a continuous map $f : \Sigma \setminus \{z_2, \cdots, z_{N_p}\} \to M$ is called a **stable** (J,H)**-broken solution with a joint** $\tilde{y}$ **and of class** D if we may divide $(\Sigma, \underline{l})$ into two semistable $\mathcal{F}$-curve $(\Sigma_+, \underline{l}^+)$ and $(\Sigma_-, \underline{l}^-)$ at some double point z_{i_0} between two principal components, $2 \le i_0 \le N_p$, such that $f|_{\Sigma_+}$ and $f|_{\Sigma_-}$ are the stable $(J,H)_+$-disk with cap $\tilde{y}$ and stable $(J,H)_-$-disk with cap $\tilde{y}\sharp(-D)$ respectively. Furthermore, a tuple $(f, \Sigma, \underline{l})$ is called a **stable** (J,H)**-broken solution of class** D if it is stable (J,H)-broken solution with a joint $\tilde{y}$ and of class D for some $\tilde{y} \in \tilde{\mathcal{P}}(H)$.

It is easily seen that if a stable (J,H)-broken solution $(f, \Sigma, \underline{l})$ with a joint $\tilde{y}$ and of a class D has at least three principal components then it is also such a broken solution with another joint different from $\tilde{y}$ and of class D. As usual we may define the equivalence class of such a broken solution in an obvious way. But it should be noted that two equivalence classes of a given stable (J,H)-broken solution $(f, \Sigma, \underline{l})$ with a joint $\tilde{y}$ as a stable (J,H)-broken solution with a joint $\tilde{y}$ and as a stable (J,H)-broken solution are same. We still denote by $\langle f \rangle$ the equivalence class of $(f, \Sigma, \underline{l})$. Define the energy of such a $\langle f \rangle$ by $E_D(\langle f \rangle) = E_D(f) = E_+(f|_{\Sigma_+}) + E_-(f|_{\Sigma_-})$. Then (2.6) yields that $E_D(\langle f \rangle) \le \omega(D) + 2\max|H|$. Let us denote by $\overline{\mathcal{M}}_D(\tilde{y}; H, J)$ and $\overline{\mathcal{M}}_D(H,J)$ the spaces of all equivalence classes of the two kinds of maps respectively. Then

$$\overline{\mathcal{M}}_D(H,J) = \cup_{\tilde{y} \in \tilde{\mathcal{P}}(H)} \overline{\mathcal{M}}_D(\tilde{y}; H, J). \tag{4.12}$$

However, one should also note that $\overline{\mathcal{M}}_D(\tilde{y}; H, J)$ and $\overline{\mathcal{M}}_D(\tilde{z}; H, J)$ have probably nonempty intersection for two different $\tilde{y}$ and $\tilde{z}$. Thus we mayn't affirm the above unions to be the disjoint unions. But each stable (J,H)-broken solution has at most $2n = \dim M$ joints. As in (4.3) and (4.4) we can prove the union at the right side of (4.12) is actually a finite unions, i.e., other $\overline{\mathcal{M}}_D(\tilde{y}; H, J)$ are empty except finitely many $\tilde{y} \in \tilde{\mathcal{P}}(H)$, saying $\tilde{y}_1, \cdots, \tilde{y}_t, t \ge r$ because of (4.10) and the fact that $\mathcal{M}_D(\tilde{y}; H, J) \subset \overline{\mathcal{M}}_D(\tilde{y}; H, J)$. Then we get that

$$\overline{\mathcal{M}}_D(H,J) = \cup_{i=1}^t \overline{\mathcal{M}}_D(\tilde{y}_i; H, J). \tag{4.13}$$

Note that by the above assumptions if $t > r$ then $\mathcal{M}_D(\tilde{y}_i; H, J) = \emptyset$ for $r < i \le t$. Unlike in (4.10) the union at the right side is not necessarily disjoint. However, as in Proposition 2.5 one easily shows that the spaces $\overline{\mathcal{M}}_D(\tilde{y}; H, J)$ and $\overline{\mathcal{M}}_D(H, J)$ equipped with the weak C^∞-topology are the Hausdorff compactifications of $\mathcal{M}_D(\tilde{y}; H, J)$ and $\mathcal{M}_D(H, J)$ respectively (cf.[LiuT1]). Moreover, the notions of the corresponding stable L^p_k-broken maps may also be introduced in the similar way. Let $\overline{\mathcal{M}}^\nu_D(\tilde{y}; H, J)$ and $\overline{\mathcal{M}}^\nu_D(H, J)$ be the corresponding virtual moduli cycles to them respectively. Consider the evaluation maps

$$\mathrm{EV}^\nu_D(\tilde{y}) : \overline{\mathcal{M}}^\nu_D(\tilde{y}; H, J) \to M \times M \quad \text{and} \quad \mathrm{EV}^\nu_D : \overline{\mathcal{M}}^\nu_D(H, J) \to M \times M$$

given by $\mathrm{EV}^\nu_D(\tilde{y})(f) = (f(z_-), f(z_+))$ and $\mathrm{EV}^\nu_D(f) = (f(z_-), f(z_+))$, where z_- and z_+ two double points on the chain of the principal components of the domain of f (cf. Def. 2.1). For the map $\bar{E}^u_a \times \bar{E}^s_b$ in (4.7), if

$$\mu(a) - \mu(b) + 2c_1(D) = 0, \tag{4.14}$$

then for a generic $(h, g) \in \mathcal{O}(h_0) \times \mathcal{R}$ we get two (rational) intersection numbers

$$\begin{aligned} &n^\nu_D(a, b, \tilde{y}) := (\bar{E}^u_a \times \bar{E}^s_b) \cdot \mathrm{EV}^\nu_D(\tilde{y}) \quad \text{and} \\ &n^\nu_D(a, b) := (\bar{E}^u_a \times \bar{E}^s_b) \cdot \mathrm{EV}^\nu_D, \end{aligned} \tag{4.15}$$

which are independent of generic ν and $(h, g) \in \mathcal{O}(h_0) \times \mathcal{R}$.

Proposition 4.3. *Under the above assumptions it holds that*

$$n^\nu_D(a, b, \tilde{y}) = n^{\nu^+}_+(a, \tilde{y}) \cdot n^{\nu^-}_-(b, \tilde{y}\sharp(-D)), \tag{4.16}$$

$$n^\nu_D(a, b) = \textstyle\sum_{i=1}^r n^\nu_D(a, b, \tilde{y}_i). \tag{4.17}$$

From these it easily follows that (4.5) and (4.9) become

$$m^\nu_{+,-}(\langle a, A\rangle; \langle b, B\rangle) = (\bar{E}^u_a \times \bar{E}^s_b) \cdot \mathrm{EV}^\nu_{B-A}. \tag{4.18}$$

Proof of Proposition 4.3. We first prove (4.16). By the definition in (3.2) the rational numbers $n^{\nu^+}_+(a, \tilde{y})$ and $n^{\nu^-}_-(b, \tilde{y}\sharp(-D))$ are independent of the choices of generic small ν^+ and ν^-. Therefore, we only need to prove (4.16) for suitable regular ν and $\nu^\pm$. Notice that $\overline{\mathcal{M}}_D(\tilde{y}; H, J)$ may be identified with the product

$$\overline{\mathcal{M}}_+(\tilde{y}; H, J) \times \overline{\mathcal{M}}_-(\tilde{y}\sharp(-D); H, J)$$

by the map $\langle f, \Sigma, \underline{l}\rangle \mapsto (\langle f|_{\Sigma_+}, \Sigma_+, \underline{l}^+\rangle, \langle f|_{\Sigma_-}, \Sigma_-, \underline{l}^-\rangle)$. More generally, let

$$\mathcal{B}^{p,k}_D(\tilde{y}, ; H), \quad \mathcal{B}^{p,k}_+(\tilde{y}; H) \quad \text{and} \quad \mathcal{B}^{p,k}_-(\tilde{y}\sharp(-D); H)$$

be the corresponding L^p_k-stable map spaces then the first one can be identified with the product space $\mathcal{B}^{p,k}_+(\tilde{y}; H) \times \mathcal{B}^{p,k}_-(\tilde{y}\sharp(-D); H)$ by the natural map

$$\mathrm{GL} : (\langle f_+, \Sigma_+, \underline{l}^+\rangle, \langle f_-, \Sigma_-, \underline{l}^-\rangle) \mapsto \langle f_+ \cup_y f_-, \Sigma_+ \cup \Sigma_-, \underline{l}^+ \cup \bar{l}^-\rangle. \tag{4.19}$$

We shall prove that a virtual moduli cycle $\overline{\mathcal{M}}^{\nu}_D(\tilde{y};H,J)$ for $\overline{\mathcal{M}}_D(\tilde{y};H,J)$ naturally induces the virtual moduli cycle $\overline{\mathcal{M}}^{\nu^+}_+(\tilde{y};H,J)$ for $\overline{\mathcal{M}}_+(\tilde{y};H,J)$ and that $\overline{\mathcal{M}}^{\nu^-}_-(\tilde{y}\sharp(-D);H,J)$ for $\overline{\mathcal{M}}_-(\tilde{y}\sharp(-D);H,J)$ such that

$$\overline{\mathcal{M}}^{\nu}_D(\tilde{y};H,J)= \{f_+\sharp_y f_-|(f_+,f_-)\in \overline{\mathcal{M}}^{\nu^+}_+(\tilde{y};H,J)\times \overline{\mathcal{M}}^{\nu^-}_-(\tilde{y}\sharp(-D);H,J)\}. \tag{4.20}$$

Once it is proved. Note that for $f=f_+\sharp_y f_-\equiv (f_+,f_-)\in \overline{\mathcal{M}}^{\nu}_D(\tilde{y};H,J)$,

$$\mathrm{EV}^{\nu}_D(\tilde{y})(f)=(\mathrm{EV}^{\nu^+}_+(\tilde{y})(f_+),\ \mathrm{EV}^{\nu^-}_-(\tilde{y}\sharp(-D))(f_-)).$$

By (4.15) and (4.20), for a generic choice of $(h,g)\in\mathcal{O}(h_0)\times\mathcal{R}$

$$\begin{aligned}
n^{\nu}_D(a,b,\tilde{y}) &= (\bar{E}^u_a\times\bar{E}^s_b)\cdot \mathrm{EV}^{\nu}_D(\tilde{y})\\
&= (\bar{E}^u_a\times\bar{E}^s_b)\cdot (\mathrm{EV}^{\nu^+}_+(\tilde{y})\times \mathrm{EV}^{\nu^-}_-(\tilde{y}\sharp(-D)))\\
&= (\bar{E}^u_a\cdot \mathrm{EV}^{\nu^+}_+(\tilde{y}))\times (\bar{E}^s_b\cdot \mathrm{EV}^{\nu^-}_-(\tilde{y}\sharp(-D)))\\
&= n^{\nu^+}_+(a,\tilde{y})\cdot n^{\nu^-}_-(b,\tilde{y}\sharp(-D)).
\end{aligned}$$

Namely, (4.16) holds. In order to prove (4.20) we follow [LiuT1, LiuT2] and [LiuT3] to choose finitely many points $\langle f^+_i\rangle=\langle f^+_i,\Sigma^+_i,\bar{l}^+_i\rangle\in\overline{\mathcal{M}}_+(\tilde{y};H,J)$ and $\langle f^-_j\rangle=\langle f^-_j,\Sigma^-_j,\bar{l}^-_j\rangle\in\overline{\mathcal{M}}_-(\tilde{y}\sharp(-D);H,J)$, and their open neighborhoods $W^+_i=W^+_i(\langle f^+_i\rangle)$ in $\mathcal{B}^{p,k}_+(\tilde{y};H)$ and those $W^-_j=W^-_j(\langle f^-_j\rangle)$ in $\mathcal{B}^{p,k}_-(\tilde{y}\sharp(-D);H)$, $i=1,\cdots,m_+$, $j=1,\cdots,m_-$, such that the following hold:

(i) $\mathcal{W}^+(\tilde{y}):=\cup_{i=1}^{m_+}W^+_i$ and $\mathcal{W}^-(\tilde{y}\sharp(-D)):=\cup_{j=1}^{m_-}W^-_j$ are respectively open neighborhoods of $\overline{\mathcal{M}}_+(\tilde{y};H,J)$ in $\mathcal{B}^{p,k}_+(\tilde{y};H)$ and those of $\overline{\mathcal{M}}_-(\tilde{y}\sharp(-D);H,J)$ in $\mathcal{B}^{p,k}_-(\tilde{y}\sharp(-D);H)$.

(ii) W^+_i and W^-_j have the uniformizers $(\widetilde{W}^+_i,\Gamma^+_i,\pi^+_i)$ and $(\widetilde{W}^-_j,\Gamma^-_j,\pi^-_j)$ respectively, $i=1,\cdots,m_+$, $j=1,\cdots,m_-$.

(iii) $\{(\widetilde{W}^+_i,\Gamma^+_i,\pi^+_i)\}_{i=1}^{m_+}$ and $\{(\widetilde{W}^-_j,\Gamma^-_j,\pi^-_j)\}_{j=1}^{m_-}$ constitute the orbifold atlases on $\mathcal{W}^+(\tilde{y})$ and $\mathcal{W}^-(\tilde{y}\sharp(-D))$ respectively.

(iv) There exist the local orbifold bundles $\widetilde{\mathcal{L}}^+_i\to\widetilde{W}^+_i$ and $\widetilde{\mathcal{L}}^-_j\to\widetilde{W}^-_j$ with uniformizing groups Γ^+_i and Γ^-_j respectively.

(v) There exist two obvious smooth sections

$$\begin{aligned}
&\bar{\partial}^+_{J,H}:\mathcal{W}^+(\tilde{y})\to\mathcal{L}^+=\cup_{i=1}^{m_+}\mathcal{L}^+_i\ \ \text{and}\\
&\bar{\partial}^-_{J,H}:\mathcal{W}^-(\tilde{y}\sharp(-D))\to\mathcal{L}^-=\cup_{j=1}^{m_-}\mathcal{L}^-_j
\end{aligned}$$

such that their zero sets are $\overline{\mathcal{M}}_+(\tilde{y};H,J)$ and $\overline{\mathcal{M}}_-(\tilde{y}\sharp(-D);H,J)$, respectively, and that they may be lifted to the collections $\{\bar{\partial}^{+,i}_{J,H}\}_{i=1}^{m_+}$ of Γ^+_i-equivariant sections of $\widetilde{\mathcal{L}}^+_i\to\widetilde{W}^+_i$ and those $\{\bar{\partial}^{-,j}_{J,H}\}_{j=1}^{m_-}$ of Γ^-_j-equivariant sections of $\widetilde{\mathcal{L}}^-_j\to\widetilde{W}^-_j$ respectively.

As defined above (2.13) we have corresponding index sets

$$\mathcal{N}^{\pm} := \{I \subset \{1, \cdots, m^{\pm}\} \,|\, W_I^{\pm} = \cap_{k\in I} W_k^{\pm} \neq \emptyset\}.$$

For $I = \{i_1, \cdots, i_p\} \in \mathcal{N}^+$ and $J = \{j_1, \cdots, j_q\} \in \mathcal{N}^-$ we denote by $\Gamma_I^+ = \prod_{k=1}^p \Gamma_{i_k}^+$ and $\Gamma_J^- = \prod_{l=1}^q \Gamma_{j_l}^-$. Correspondingly, we have also the fibre products

$$\widetilde{W}_I^{\Gamma_I^+} = \{x_I^+ = (x_i^+)_{i\in I} \,|\, \pi_i^+(x_i^+) = \pi_{i'}^+(x_{i'}) \in W_I^+,\ \forall i, i' \in I\},$$
$$\widetilde{W}_J^{\Gamma_J^-} = \{y_J^- = (y_j^-)_{j\in J} \,|\, \pi_j^-(y_i^-) = \pi_{j'}^-(y_{j'}) \in W_J^-,\ \forall j, j' \in J\}$$

and the projections $\pi_I^+ : \widetilde{W}_I^{\Gamma_I^+} \to W_I^+$ and $\pi_J^- : \widetilde{W}_J^{\Gamma_J^-} \to W_J^-$. For $I, I' \in \mathcal{N}^+$ with $I \subset I'$ and $J, J' \in \mathcal{N}^-$ with $J \subset J'$ we have also obvious projections $\pi_{I,I'}^+$ and $\pi_{J,J'}^-$ satisfying $\pi_{I'}^+ \circ \pi_{I,I'}^+ = \pi_I^+$ and $\pi_{J'}^- \circ \pi_{J,J'}^- = \pi_J^-$. Following [LiuT3] we may construct the desingularizations of $\widetilde{W}^{\Gamma^+}$ and $\widetilde{W}^{\Gamma^-}$ as follows:

$$\widehat{W}^{\Gamma^+} = \left\{\widehat{W}_I^{\Gamma^+}, \pi_{I,I'}^+ \,|\, I \subset I' \in \mathcal{N}^+\right\} \text{ and}$$
$$\widehat{W}^{\Gamma^-} = \left\{\widehat{W}_J^{\Gamma^-}, \pi_{J,J'}^+ \,|\, J \subset J' \in \mathcal{N}^-\right\}.$$

By (2.15) and (2.16), for every $I \subset \mathcal{N}^+$ (resp. $J \subset \mathcal{N}^-$) there exist subsets $V_I^+ \subset W_I^+$ (resp. $V_I^- \subset W_I^-$) such that:
(a) $\overline{\mathcal{M}}_+(\tilde{y}; H, J) \subset \cup_{I\in\mathcal{N}^+} V_I^+$ (resp. $\overline{\mathcal{M}}_-(\tilde{y}\sharp(-D); H, J) \subset \cup_{\in\mathcal{N}^-} V_J^-$);
(b) $V_{I'}^+ \cap V_I^+ \neq \emptyset$ (resp. $V_{J'}^- \cap V_J^- \neq \emptyset$) only when $I \subset I'$ (resp. $J \subset J'$).
Then one can obtain from $\widetilde{W}_I^{\Gamma_I^+}$ (resp. $\widetilde{W}_J^{\Gamma_J^-}$) the desingularization $\widehat{V}_I^{\Gamma_I^+}$ (resp. $\widehat{V}_J^{\Gamma_J^-}$) of the restriction of $\widetilde{W}_I^{\Gamma_I^+}$ (resp. $\widetilde{W}_J^{\Gamma_J^-}$) to V_I^+ (resp. V_J^-). Let us denote by

$$\widehat{V}^{\Gamma^+} = \left\{\widehat{V}_I^{\Gamma_I^+}, \pi_{I,I'}^+ \;\middle|\; I \subset I' \in \mathcal{N}^+\right\} \text{ and}$$
$$\widehat{V}^{\Gamma^-} = \left\{\widehat{V}_J^{\Gamma_J^-}, \pi_{J,J'}^+ \;\middle|\; J \subset J' \in \mathcal{N}^-\right\}.$$

With the same way we may use $\widetilde{\mathcal{L}}_i^+$ and $\widetilde{\mathcal{L}}_j^-$ to construct $\widetilde{\mathcal{L}}_I^{\Gamma_I^+}$, $\widehat{\mathcal{L}}_I^{\Gamma_I^+}$ and $\widetilde{\mathcal{L}}_J^{\Gamma_J^-}$, $\widehat{\mathcal{L}}_J^{\Gamma_J^-}$ for $I \in \mathcal{N}^+$ and $J \in \mathcal{N}^-$, and obtain the following systems of the bundles

$$(\widetilde{\mathcal{L}}^{\Gamma^+}, \widetilde{W}^{\Gamma^+}) = \left\{\left(\widetilde{\mathcal{L}}^{\Gamma_I^+}, \widetilde{W}_I^{\Gamma_I^+}\right) \;\middle|\; I \in \mathcal{N}^+\right\} \quad \text{and}$$
$$(\widehat{\mathcal{L}}^{\Gamma^+}, \widehat{V}^{\Gamma^+}) = \left\{\left(\widehat{\mathcal{L}}^{\Gamma_I^+}, \widehat{V}_I^{\Gamma_I^+}\right) \;\middle|\; I \in \mathcal{N}^+\right\},$$
$$(\widetilde{\mathcal{L}}^{\Gamma^-}, \widetilde{W}^{\Gamma^-}) = \left\{\left(\widetilde{\mathcal{L}}^{\Gamma_J^-}, \widetilde{W}_J^{\Gamma_J^-}\right) \;\middle|\; J \in \mathcal{N}^-\right\} \quad \text{and}$$
$$(\widehat{\mathcal{L}}^{\Gamma^+}, \widehat{V}^{\Gamma^-}) = \left\{\left(\widehat{\mathcal{L}}_J^{\Gamma_J^+}, \widehat{V}_J^{\Gamma_J^-}\right) \;\middle|\; J \in \mathcal{N}^-\right\}.$$

In particular, $(\widehat{\mathcal{L}}^{\Gamma^+}, \widehat{V}^{\Gamma^+})$ and $(\widehat{\mathcal{L}}^{\Gamma^-}, \widehat{V}^{\Gamma^-})$ are also the systems of the stratified Banach bundles. For $i = 1, \cdots, m_+$, $j = 1, \cdots, m_-$, let $\langle f_{ij}\rangle = \langle f_i^+\sharp_y f_j^-\rangle$ and

$$\begin{aligned} W_{ij} &= \{\langle g^+\sharp_y g^-\rangle \,|\, \langle g^+\rangle \in W_i^+,\ \langle g^-\rangle \in W_j^-\} \ \text{ and} \\ \widetilde{W}_{ij} &= \{g^+\sharp_y g^- \,|\, g^+ \in \widetilde{W}_i^+,\ g^- \in \widetilde{W}_j^-\}. \end{aligned}$$

Then $\langle f_{ij}\rangle \in \overline{\mathcal{M}}_D(\tilde{y}; H, J)$ and $\{W_{ij}|1 \le i \le m_+, 1 \le j \le m_-\}$ is a covering of $\overline{\mathcal{M}}_D(\tilde{y}; H, J)$. Moreover, each W_{ij} has a uniformizer $(\widetilde{W}_{ij}, \Gamma_{ij}, \pi_{ij})$ with

$$\Gamma_{ij} = \Gamma_i^+ \times \Gamma_j^- \quad \text{and} \quad \pi_{ij}(g^+\sharp_y g^-) = \mathrm{GL}(\pi_i^+(g^+), \pi_j^-(g^-)).$$

The action of Γ_{ij} on $\widetilde{W}_{ij}$ is given by $(\phi^+, \phi^-)\cdot(g^+, g^-) = (\phi^+\cdot g^+, \phi^-\cdot g^-)$. As in §2 we may construct the local orbifold bundle $\mathcal{L}_{ij} \to W_{ij}$ uniformized by $\widetilde{\mathcal{L}}_{ij} \to \widetilde{W}_{ij}$ with uniformizing group Γ_{ij}. Moreover, there exists a natural section $\bar{\partial}^D_{H,J} : \mathcal{W} \to \mathcal{L} = \cup_{i,j} W_{ij}$ whose zero set is $\overline{\mathcal{M}}_D(\tilde{y}; H, J)$ and that lifts to a collection $\{\bar{\partial}^{D,ij}_{J,H}|1 \le i \le m_+, 1 \le j \le m_-\}$ of Γ_{ij}-equivariant sections of $\widetilde{\mathcal{L}}_{ij} \to \widetilde{W}_{ij}$. Here $\bar{\partial}^{D,ij}_{J,H}$, $\bar{\partial}^{+,i}_{J,H}$ and $\bar{\partial}^{-,j}_{J,H}$ satisfy: for $g = g^+\sharp_y g^- \in \widetilde{W}_{ij}$ and $\xi \in (\widetilde{\mathcal{L}}_{ij})_g = L^p_{k-1}(\wedge^{0,1}(g^*TM))$ it must hold that

$$\bar{\partial}^{D,ij}_{J,H}\xi = \big(\bar{\partial}^{+,i}_{J,H}(\xi|_{\Sigma^+})\big)\sharp\big(\bar{\partial}^{-,j}_{J,H}(\xi|_{\Sigma^-})\big)$$

because two terms at the right side are same when restricted to $\Sigma^+\cap\Sigma^-$. Here Σ^+ and Σ^- are the domains of g^+ and g^- respectively. For given $f_+ \in \mathcal{B}^{p,k}_+(\tilde{y}; H)$ and $f_- \in \mathcal{B}^{p,k}_-(\tilde{y}\sharp(-D); H)$ the elements $\xi^+ \in L^p_{k-1}(\wedge^{0,1}(f_+^*TM))$ and $\xi^- \in L^p_{k-1}(\wedge^{0,1}(f_-^*TM))$ might not be glued into one of $L^p_{k-1}(\wedge^{0,1}((f_+\sharp_y f_-)^*TM))$. Thus we might not glue $\mathcal{L}^+$ and $\mathcal{L}^-$ in general.

Also denote by $\mathcal{N} = \mathcal{N}^+ \times \mathcal{N}^-$. For (I, J) and (I', J') in $\mathcal{N}$ we say $(I, J) \subset (I', J')$ if $I \subset I'$ and $J \subset J'$. Corresponding to this covering we have $W_{(I,J)} = \mathrm{GL}(W_I \times W_J)$, $\Gamma_{(I,J)} = \Gamma_I^+ \oplus \Gamma_J^-$ and the system of bundles

$$\begin{aligned} (\widetilde{\mathcal{L}}^\Gamma, \widetilde{W}^\Gamma) &= \Big\{\Big(\widetilde{\mathcal{L}}^{\Gamma_{(I,J)}}_{(I,J)}, \widetilde{W}^{\Gamma_{(I,J)}}_{(I,J)}\Big), \pi_{(I,J)}, \pi^{(I',J')}_{(I,J)} \ \Big|\ (I,J) \subset (I',J') \in \mathcal{N}\Big\}, \\ (\widehat{\mathcal{L}}^\Gamma, \widehat{V}^\Gamma) &= \Big\{\Big(\widehat{\mathcal{L}}^{\Gamma_{(I,J)}}_{(I,J)}, \widehat{V}^{\Gamma_{(I,J)}}_{(I,J)}\Big), \pi_{(I,J)}, \pi^{(I',J')}_{(I,J)} \ \Big|\ (I,J) \subset (I',J') \in \mathcal{N}\Big\}. \end{aligned}$$

Let $f_{ij} = f_i^+\sharp_y f_j^-$ and $R(f_{ij})$ be the cokernel of $D(\bar{\partial}^{D,ij}_{J,H})(f_{ij})$ in $(\widetilde{\mathcal{L}}_{ij})_{f_{ij}}$ as before. Take smooth cut-off functions $\beta_\epsilon(f_i^+)$ on the domain Σ_i^+ of f_i^+ and $\beta_\epsilon(f_j^-)$ on that Σ_j^- of f_j^- supported outside of the ϵ-neighborhood of their double points. Then $\beta_\epsilon(f_i^+)$ and $\beta_\epsilon(f_j^-)$ naturally determine a smooth cut-off function, denoted by $\beta_\epsilon(f_{ij}) = \beta_\epsilon(f_i^+)\sharp\beta_\epsilon(f_j^-)$, on the domain $\Sigma_{ij} = \Sigma_i^+\sharp\Sigma_j^-$ of f_{ij} supported outside of the ϵ-neighborhood of their double points. As in Step 1 in §2.2 and (2.19) we may use these to define corresponding spaces

$$R_\epsilon(f_{ij}) \quad \text{and} \quad R^\epsilon(\{f_{ij}\}) = \oplus_{i=1}^{m_+}\oplus_{j=1}^{m_-} R_\epsilon(f_{ij}).$$

Now as in (2.18) we take the smooth Γ_i^+-invariant cut-off functions $\gamma(f_i^+)$ on $\widetilde{W}_i^+$ and Γ_j^--invariant cut-off functions $\gamma(f_j^-)$ on $\widetilde{W}_j^-$ such that for each $\nu_{ij}\in R_\epsilon(f_{ij})$ (which may be written as $\nu_i^+\sharp\nu_j^-$ for some $\nu_i^+\in R_\epsilon(f_i^+)$ with domain Σ_i^+ and $\nu_j^-\in R_\epsilon(f_j^-)$ with domain Σ_j^-),

$$\gamma(f_i^+)\cdot\widetilde{\nu_{ij}|_{\Sigma_i^+}}=\gamma(f_i^+)\cdot\widetilde{\nu_i^+}\quad\text{and}\quad\gamma(f_j^-)\cdot\widetilde{\nu_{ij}|_{\Sigma_j^-}}=\gamma(f_j^-)\cdot\widetilde{\nu_j^-}$$

will give rise to the global section

$$\overline{\nu_i^+}=\overline{\nu_{ij}|_{\Sigma_i^+}}=\Big\{\big(\overline{\nu_{ij}|_{\Sigma_i^+}}\big)_I=\big(\overline{\nu_i^+}\big)_I\,|\,I\in\mathcal{N}^+\Big\}$$

of $(\widehat{\mathcal{L}}^+,\widehat{V}^+)$ and that

$$\overline{\nu_j^-}=\overline{\nu_{ij}|_{\Sigma_j^-}}=\Big\{\big(\overline{\nu_{ij}|_{\Sigma_j^-}}\big)_J=\big(\overline{\nu_j^-}\big)_J\,|\,J\in\mathcal{N}^-\Big\}$$

of $(\widehat{\mathcal{L}}^-,\widehat{V}^-)$ respectively. Then $\gamma(f_i^+)\sharp\gamma(f_j^-)\cdot\widetilde{\nu_{ij}}$ (defined by $(\gamma(f_i^+)\sharp\gamma(f_j^-)\cdot\widetilde{\nu_{ij}})(g_i^+\sharp g_j^-)=\gamma(f_i^+)(g_i^+)\cdot\gamma(f_j^-)(g_j^-)\cdot\widetilde{\nu_{ij}}(g_i^+\sharp g_j^-)$) can yield a global section $\overline{\nu_{ij}}=\{(\overline{\nu_{ij}})_{I,J}\,|\,(I,J)\in\mathcal{N}\}$ of $(\widehat{\mathcal{L}},\widehat{V})$. For $\delta>0$ small enough, it follows as before that for a generic choice of $\nu\in R^\epsilon_\delta(\{f_{ij}\})$ the global section transversal to the zero section

$$S^{D,\nu}_{J,H}=\big\{S_D^{\nu_{(I,J)}}=\bar\partial^D_{J,H}+\bar\nu_{(I,J)}\;\big|\;(I,J)\in\mathcal{N}\big\},$$

and thus get the virtual moduli cycle

$$\overline{\mathcal{M}}^\nu_D(\tilde y;H,J)=\textstyle\sum_{(I,J)\in\mathcal{N}}\frac{1}{|\Gamma_{(I,J)}|}\big\{\pi_{(I,J)}:\mathcal{M}_D^{\nu_{(I,J)}}(\tilde y;H,J)\to\mathcal{W}\big\}.$$

Here $\mathcal{M}_D^{\nu_{(I,J)}}(\tilde y;H,J)=(S_D^{\nu_{(I,J)}})^{-1}(0)$. Note that ν can be expressed as $\oplus_{i=1}^{m_+}\oplus_{j=1}^{m_-}$ ν_{ij} with $\nu_{ij}=\nu_i^+\sharp\nu_j^-\in R_\epsilon(f_{ij})$, $i=1,\cdots,m_+$ and $j=1,\cdots,m_-$. We put

$$\bar\nu^+=\sum_{i=1}^{m_+}\overline{\nu_{ij}|_{\Sigma_i^+}}=\sum_{i=1}^{m_+}\overline{\nu_i^+}\quad\text{and}\quad\bar\nu^-=\sum_{j=1}^{m_-}\overline{\nu_{ij}|_{\Sigma_j^-}}=\sum_{j=1}^{m_-}\overline{\nu_j^-}.\tag{4.21}$$

They are the global section of $(\widehat{\mathcal{L}}^+,\widehat{V}^+)$ and that of $(\widehat{\mathcal{L}}^-,\widehat{V}^-)$ respectively. We assert that $\bar\partial^+_{J,H}+\bar\nu^+$ (resp. $\bar\partial^-_{J,H}+\bar\nu^-$) is also transversal to the zero section of $(\widehat{\mathcal{L}}^+,\widehat{\mathcal{V}}^+)$ (resp.$(\widehat{\mathcal{L}}^-,\widehat{\mathcal{V}}^-)$). We only prove the assertion for $\bar\partial^+_{J,H}+\bar\nu^+$. Firstly, using (4.19) it is not hard to check that $f\in\mathcal{M}_D^{\nu_{(I,J)}}(\tilde y;H,J)$ if and only if $f=f^+\sharp_y f^-$ for some $f^+\in(\bar\partial^+_{J,H}+\bar\nu^+_I)^{-1}(0)$ and $f^-\in(\bar\partial^-_{J,H}+\bar\nu^-_J)^{-1}(0)$. Next, by (4.21) we have

$$\bar\nu^+=\sum_{i=1}^{m_+}\overline{\nu_{ij}|_{\Sigma_i^+}}=\Big\{\sum_{i=1}^{m_+}(\overline{\nu_{ij}|_{\Sigma_i^+}})_I\;\Big|\;I\in\mathcal{N}^+\Big\}=\Big\{\sum_{i=1}^{m_+}(\overline{\nu_i^+})_I\;\Big|\;I\in\mathcal{N}^+\Big\}.$$

For a given $f^+\in(\bar\partial^+_{J,H}+\bar\nu^+_I)^{-1}(0)$ we choose any $f^-\in(\bar\partial^-_{J,H}+\bar\nu^-_J)^{-1}(0)$ and obtain a $f=f^+\sharp_y f^-\in\mathcal{M}_D^{\nu_{(I,J)}}(\tilde y;H,J)$. Note that we can always extend any

$\xi^+ \in (\widehat{\mathcal{L}}_I^+)_{f^+}$ into an element $\xi \in (\widehat{\mathcal{L}}_{(I,J)})_f$. Since $\bar{\partial}^D_{J,H} + \bar{\nu}_{(I,J)} : \widehat{V}_{(I,J)} \to \widehat{\mathcal{L}}_{(I,J)}$ is transversal to the zero section we have $\eta \in T_f\widehat{V}_{(I,J)}$ such that $D(\bar{\partial}^D_{J,H} + \bar{\nu}_{(I,J)})(\eta) = \xi$. This implies that $D(\bar{\partial}^+_{J,H} + \bar{\nu}^+_I)(\eta|_{\Sigma^+}) = \xi^+$. The assertion is proved. In particular we get two virtual moduli cycles

$$\overline{\mathcal{M}}_+^{\nu^+}(\tilde{y}; H, J) = \sum_{I\in\mathcal{N}^+} \frac{1}{|\Gamma_I^+|}\{\pi_I^+ : \mathcal{M}_+^{\nu_I^+}(\tilde{y}; H, J) \to \mathcal{W}^+\},$$
$$\overline{\mathcal{M}}_-^{\nu^-}(\tilde{y}\sharp(-D); H, J) = \sum_{J\in\mathcal{N}^-} \frac{1}{|\Gamma_J^-|}\{\pi_J^- : \mathcal{M}_-^{\nu_J^-}(\tilde{y}\sharp(-D); H, J) \to \mathcal{W}^-\}.$$

Here $\mathcal{M}_+^{\nu_I^+}(\tilde{y}; H, J) = (\bar{\partial}^+_{J,H} + \bar{\nu}^+_I)^{-1}(0)$ and $\mathcal{M}_-^{\nu_J^-}(\tilde{y}\sharp(-D); H, J) = (\bar{\partial}^-_{J,H} + \bar{\nu}^-_J)^{-1}(0)$. Now (4.19) and the facts that $\pi_{(I,J)} = \pi_I^+ \times \pi_J^-$ and $|\Gamma_{I,J}| = |\Gamma_I^+ \times \Gamma_J^-| = |\Gamma_I^+||\Gamma_J^-|$ together lead to

$$\overline{\mathcal{M}}^\nu_D(\tilde{y}; H, J) = \overline{\mathcal{M}}_+^{\nu^+}(\tilde{y}; H, J) \times \overline{\mathcal{M}}_-^{\nu^-}(\tilde{y}\sharp(-D); H, J).$$

This is equivalent to (4.20).

Next we prove (4.17). The ideas are similar to the proof of (4.16). Denote by $\tilde{y}_i = [y_i, w_i]$. By (4.13), for $i = 1, \cdots, t$, we take small neighborhoods $W_l^{(i)}$ centred at $\langle f_l^{(i)}\rangle \in \overline{\mathcal{M}}^\nu_D(\tilde{y}_i; H, J)$ in $\mathcal{B}^{p,k}_D(\tilde{y}_i; H)$, $l = l_{i-1}+1, \cdots, l_{i-1}+l_i$ with $l_0 = 1$, such that $\{W_l^{(i)}\}_{l=l_{i-1}+1}^{l_i}$ constitutes a finite covering of $\overline{\mathcal{M}}^\nu_D(\tilde{y}_i; H, J)$ satisfying the requirements to construct the virtual moduli cycle. Let $\mathcal{L}_l^{(i)} \to W_l^{(i)}$ be the local orbifold bundle uniformized by $\widetilde{\mathcal{L}}_l^{(i)} \to \widetilde{W}_l^{(i)}$ with uniformizing group $\Gamma_l^{(i)}$ and projection $\pi_l^{(i)}$, and

$$\bar{\partial}^{(i)}_{J,H,D} : \mathcal{W}^{(i)} = \bigcup_{l=l_{i-1}+1}^{l_i} W^{(i)} \to \mathcal{L}^{(i)} = \bigcup_{l=l_{i-1}+1}^{l_i} \mathcal{L}_l^{(i)},$$
$$\bar{\partial}^D_{J,H} : \mathcal{W} = \bigcup_{i=1}^t \bigcup_{l=l_{i-1}+1}^{l_i} W^{(i)} \to \mathcal{L} = \bigcup_{i=1}^t \bigcup_{l=l_{i-1}+1}^{l_i} \mathcal{L}_l^{(i)}$$

the obvious sections whose zero sets are $\overline{\mathcal{M}}^\nu_D(\tilde{y}_i; H, J)$ and $\overline{\mathcal{M}}^\nu_D(H, J)$ respectively. Let $\mathcal{N}^{(i)}$ be the nerve of the covering $\{W_l^{(i)}\}_{l=l_{i-1}+1}^{l_i}$ of $\overline{\mathcal{M}}^\nu_D(\tilde{y}_i; H, J)$, $i = 1, \cdots, t$, and $\mathcal{N}$ be that of the covering $\cup_{i=1}^t\{W_l^{(i)}\}_{l=l_{i-1}+1}^{l_i}$ of $\overline{\mathcal{M}}^\nu_D(H, J)$ as before. The elements of $\mathcal{N}^{(i)}$ and those of $\mathcal{N}$ are denoted by $I^{(i)}$ and I respectively. Correspondingly, we have the bundle systems

$$(\widehat{\mathcal{L}}^{\Gamma^{(i)}}, \widehat{V}^{\Gamma^{(i)}}) = \left\{\left(\widehat{\mathcal{L}}^{\Gamma^{(i)}_{I^{(i)}}}_{I^{(i)}}, \widehat{V}^{\Gamma^{(i)}_{I^{(i)}}}_{I^{(i)}}\right) \,\Big|\, I^{(i)} \in \mathcal{N}^{(i)}\right\},$$
$$(\widehat{\mathcal{L}}^{\Gamma}, \widehat{V}^{\Gamma}) = \left\{(\widehat{\mathcal{L}}_I^{\Gamma_I}, \widehat{V}_I^{\Gamma_I}) \,\big|\, I \in \mathcal{N}\right\}.$$

As in (2.18) let $R^\epsilon(\{f_l^{(i)}\}) = \oplus_{i=1}^t \oplus_{l=l_{i-1}+1}^{l_i} R_\epsilon(f_l^{(i)})$ be the corresponding finite dimensional space that is used to construct the virtual moduli cycle from $(\widehat{\mathcal{L}}^\Gamma, \widehat{V}^\Gamma)$ and $\bar\partial^D_{J,H}$. Then for a generic small $\nu \in R^\epsilon(\{f_l^{(i)}\})$ we obtain a global section $\bar\nu : \widehat{V}^\Gamma \to \widehat{\mathcal{L}}^\Gamma$ such that $S^\nu_D = \{S^{\nu_I}_D = \bar\partial^D_{J,H} + \bar\nu_I \,|\, I \in \mathcal{N}\}$ is transversal to the zero section. As above we may prove that $\bar\nu$ induces a global section $\bar\nu^{(i)} : \widehat{V}^{\Gamma^{(i)}} \to \widehat{\mathcal{L}}^{\Gamma^{(i)}}$ such that

$$S_D^{\nu^{(i)}} = \left\{ S_D^{\nu^{(i)}_{I^{(i)}}} = \bar\partial^{(i)}_{J,H,D} + \bar\nu^{(i)}_{I^{(i)}} \;\middle|\; I^{(i)} \in \mathcal{N}^{(i)} \right\}$$

is also transversal to the zero section for each i. Let

$$\overline{\mathcal{M}}^\nu_D(H,J) = \sum_{I\in\mathcal{N}} \frac{1}{|\Gamma_I|}\{\pi_I : \mathcal{M}^{\nu_I}_D(H,J) \to \mathcal{W}\} \quad \text{and}$$

$$\overline{\mathcal{M}}^{\nu^{(i)}}_D(\tilde y_i; H,J) = \sum_{I^{(i)}\in\mathcal{N}^{(i)}} \frac{1}{|\Gamma^{(i)}_{I^{(i)}}|}\left\{\pi^{(i)}_{I^{(i)}} : \mathcal{M}_D^{\nu^{(i)}_{I^{(i)}}}(\tilde y_i; H,J) \to \mathcal{W}^{(i)}\right\},$$

$i = 1,\cdots,t$, be the corresponding virtual moduli cycles, where

$$\mathcal{M}^{\nu_I}_D(H,J) = (S^{\nu_I}_D)^{-1}(0) \quad \text{and} \quad \mathcal{M}_D^{\nu^{(i)}_{I^{(i)}}}(\tilde y_i; H,J) = \left(S_D^{\nu^{(i)}_{I^{(i)}}}\right)^{-1}(0).$$

By (4.10) the top strata of $\overline{\mathcal{M}}^\nu_D(H,J)$ can only contain those of $\overline{\mathcal{M}}^{\nu^{(i)}}_D(\tilde y_i; H,J)$, $i = 1,\cdots,r$. Other top strata are all empty. Thus

$$T\overline{\mathcal{M}}^\nu_D(H,J) = \cup_{i=1}^r T\overline{\mathcal{M}}^{\nu^{(i)}}_D(\tilde y_i; H,J).$$

The union is also disjoint because $\tilde y_i$, $i = 1,\cdots,r$, are different. Since (4.14) implies that the intersections in (4.15) may only occur in the top strata we arrive at

$$(\bar E^u_a \times \bar E^s_b)\cdot \mathrm{EV}^\nu_D = \sum_{i=1}^r (\bar E^u_a \times \bar E^s_b)\cdot \mathrm{EV}^{\nu^{(i)}}_D(\tilde y_i).$$

This completes the proof of (4.17). □

Step 3. Now we need to introduce a kind of deformation spaces for understanding the right side of (4.18). For $\rho \ge 0$ and $f : \mathbb{R}\times S^1 \to M$ we define $\bar\partial_{J,H_\rho} f$ by

$$\begin{aligned}\bar\partial_{J,H_\rho} f(s,t) = \partial_s f(s,t) +& \\ J(f)(\partial_t f - (\beta_+(s+\rho+1)&\cdot\beta_+(\rho+1-s))X_H(t,f)) = 0,\end{aligned} \tag{4.22}$$

where β_+ is as in §2. For such a map f we define the energy of it by

$$E_\rho(f) := \int_{-\infty}^\infty \int_0^1 |\partial_s f|^2_{g_J} ds dt. \tag{4.23}$$

By the removable singularity theorem, $E_\rho(f) < +\infty$ implies that f can be extended into a smooth map from $\mathbb{C}P^1 \approx \{-\infty\}\cup\mathbb{R}\times S^1 \cup \{\infty\}$ to M which is also J-holomorphic near 0 and ∞. For a given $D \in \Gamma$ we denote by $\mathcal{M}_D(H_\rho, J)$

the space of all maps satisfying (4.22), having finite energy and representing the class D. To compactify it we introduce:

Definition 4.4. Given $D \in \Gamma$ and a semistable $\mathcal{F}$-curve $(\Sigma, \underline{l})$ with a unique principal component(cf. Def. 2.1), a continuous map $f : \Sigma \to M$ is called a **stable** (J, H_ρ)**-map of a class** D if it represents D in the usual sense and also satisfies:

(1) On the unique principal component P with cylindrical coordinate (s,t), $f^P = f|_{P-\{z_-,z_+\}}$ satisfies: $\bar{\partial}_{J,H_\rho} f^P = 0$ and $E_\rho(f^P) < +\infty$.
(2) The restriction f_i^B of f to each bubble component B_i is J-holomorphic, and the domain of each homologically trivial bubble component is stable.

We may also define the equivalence class of such a map. Denote by $\overline{\mathcal{M}}_D(H_\rho, J)$ the space of all equivalence classes of such maps. The energy of f is defined by

$$E_\rho(f) = \int_{-\infty}^{\infty}\int_0^1 |\partial_s f^P|^2_{g_J} dsdt + \sum_i \int_{B_i} (f_i^B)^*\omega.$$

From the arguments in [Sch3, 4.2] it follows that

$$E_\rho(f) \le \omega(D) + 2\max|H|. \tag{4.24}$$

As usual, for any $\rho \ge 0$ we may construct the virtual moduli cycle $\overline{\mathcal{M}}^{\nu_\rho}_{B-A}(H_\rho, J)$ of dimension $2n + 2c_1(B-A)$ corresponding to the space $\overline{\mathcal{M}}_{B-A}(H_\rho, J)$. Consider the evaluation

$$\mathrm{E}^{\nu_\rho}_{B-A} : \overline{\mathcal{M}}^{\nu_\rho}_{B-A}(H_\rho, J) \to M \times M,\ f \mapsto (f(z_-), f(z_+)).$$

By (4.2), for a generic $(h,g) \in \mathcal{O}(h_0) \times \mathcal{R}$ we get a rational intersection number

$$n^{\nu_\rho}_{B-A}(a,b,h,g;H_\rho,J) := (\bar{E}^u_a \times \bar{E}^s_b)\cdot \mathrm{E}^{\nu_\rho}_{B-A}. \tag{4.25}$$

Define $(\Psi\circ\Phi)_\rho : QC_*(M,\omega;h,g;\mathbb{Q}) \to QC_*(M,\omega;h,g;\mathbb{Q})$ by

$$(\Psi\circ\Phi)_\rho(\langle a, A\rangle) = \sum_{\mu(\langle b,B\rangle)=\mu(\langle a,A\rangle)} n^{\nu_\rho}_{B-A}(a,b,h,g;H_\rho,J)\cdot\langle b, B\rangle.$$

Proposition 4.5. *For every $\rho \ge 0$, $(\Psi\circ\Phi)_\rho$ is a chain homomorphism.*

Proof. Firstly, note that if $n^{\nu_\rho}_{B-A}(a,b,h,g;H_\rho,J) \ne 0$ then $\overline{\mathcal{M}}_{B-A}(H_\rho,J) \ne \emptyset$ and thus it follows from (4.24) that $0 \le \omega(B-A) + 2\max|H|$. Using the fact, as in Step1 we can prove that $(\Psi\circ\Phi)_\rho$ indeed maps $QC_*(M,\omega;h,g;\mathbb{Q})$ into itself.

Next, we show that $(\Psi\circ\Phi)_\rho$ commutes with the boundary operator ∂^Q in (1.6). It suffices to prove that $\partial\circ(\Psi\circ\Phi)_\rho(\langle a,A\rangle) = (\Psi\circ\Phi)_\rho\circ\partial^Q(\langle a,A\rangle)$ for each $\langle a,A\rangle \in (\mathrm{Crit}(h)\times\Gamma)_k$. The direct computation shows that the left equals

$$\sum_{\mu(\langle c,B\rangle)=k-1}\left[\sum_{\mu(d)=\mu(c)+1} n(d,c) n^{\nu_\rho}_{B-A}(a,d,h,g;H_\rho,J)\right]\cdot\langle c,B\rangle,$$

and the right does

$$\sum_{\mu(\langle c,B\rangle)=k-1}\left[\sum_{\mu(b)=\mu(a)-1} n(a,b)n^{\nu_\rho}_{B-A}(b,c,h,g;H_\rho,J)\right]\cdot\langle c,B\rangle.$$

Therefore we only need to prove

$$\begin{aligned}&\sum_{\mu(d)=\mu(c)+1} n(d,c)n^{\nu_\rho}_{B-A}(a,d,h,g;H_\rho,J)\\ &=\sum_{\mu(b)=\mu(a)-1} n(a,b)n^{\nu_\rho}_{B-A}(b,c,h,g;H_\rho,J).\end{aligned}$$

This can be proved as Propositions 3.5 and 3.6. In fact, by

$$\mu(a)-\mu(c)+2c_1(B-A)=\mu(\langle a,A\rangle)-\mu(\langle c,B\rangle)=1,$$

for a generic $(h,g)\in\mathcal{O}(h_0)\times\mathcal{R}$ the fibre product

$$(\overline{W}^u(a,h,g)\times\overline{W}^s(c,h,g))\times_{\bar{E}^u_a\times\bar{E}^s_c=\mathrm{E}^{\nu_\rho}_{B-A}}\overline{\mathcal{M}}^{\nu_\rho}_{B-A}(H_\rho,J)$$

is a collection of compatible local cornered smooth manifolds of dimension 1 and with the natural orientations. Its boundary is given by the union

$$\begin{aligned}&(\cup_{\mu(b)=\mu(a)-1}n(a,b)\cdot(W^u(b,h,g)\times\overline{W}^s(c,h,g))\times_{R_{bc}}\overline{\mathcal{M}}^{\nu_\rho}_{B-A}(H_\rho,J))\cup\\ &(-\cup_{\mu(d)=\mu(c)+1}n(d,c)\cdot(W^u(a,h,g)\times\overline{W}^s(d,h,g))\times_{R_{ad}}\overline{\mathcal{M}}^{\nu_\rho}_{B-A}(H_\rho,J))\end{aligned}$$

because $\overline{\mathcal{M}}^{\nu_\rho}_{B-A}(H_\rho,J)$ has no components of codimension 1. Here R_{ad} and R_{bc} are representing $\bar{E}^u_a\times\bar{E}^s_d=\mathrm{E}^{\nu_\rho}_{B-A}$ and $\bar{E}^u_b\times\bar{E}^s_c=\mathrm{E}^{\nu_\rho}_{B-A}$. Hence as before the conclusions can follow from

$$\sharp\partial((\overline{W}^u(a,h,g)\times\overline{W}^s(c,h,g))\times_{\bar{E}^u_a\times\bar{E}^s_c=\mathrm{E}^{\nu_\rho}_{B-A}}\overline{\mathcal{M}}^{\nu_\rho}_{B-A}(H_\rho,J))=0.$$

□

Our purpose is to prove that $(\Psi\circ\Phi)_0$ is chain homotopy equivalent to $\Psi\circ\Phi$. To this goal let us consider the space

$$\mathcal{M}_{B-A}(\{H_\rho\},J)=\cup_{\rho\ge0}\{\rho\}\times\mathcal{M}_{B-A}(H_\rho,J).$$

From the arguments in [HS1, S1, S2] and [Sch2] it is not hard to derive that for any sequence $u^m\in\mathcal{M}_{B-A}(H_{\rho_m},J)$ with $\rho_m\to+\infty$ there must exist a subsequence (still denoted by u^m), finitely many elements $\tilde{x}_1,\cdots,\tilde{x}_k\in\tilde{\mathcal{P}}(H)$, and $u_0\in\mathcal{M}_+(\tilde{x}_1;\theta,J)$, $u_j\in\mathcal{M}(\tilde{x}_j,\tilde{x}_{j+1};H,J)$, $j=1,\cdots,k-1$, $u_k\in\mathcal{M}_-(\tilde{x}_k;H,J)$, and sequences $-\rho_m-1\equiv s^0_m<\cdots<s^k_m\equiv\rho_m+1$, such that $u^m(s+s^i_m,t)$ **converge modulo bubbling** to $u_i(s,t)$ for $i=0,\cdots,k$ (see [S2] for the precise definition of this term). Moreover, if w^i_l, $i=1,\cdots,i_l$, are all bubbles attached to u_i, then the connected sum of all u_i, w^i_l, $l=1,\cdots,i_l$, $i=0,\cdots,k$, represents the class $B-A$. This convergence result shows that the stabilized space of $\mathcal{M}_{B-A}(\{H_\rho\},J)$ is given by

$$\overline{\mathcal{M}}_{B-A}(\{H_\rho\},J)=\cup_{\rho\in[0,+\infty]}\{\rho\}\times\overline{\mathcal{M}}_{B-A}(H_\rho,J),\tag{4.26}$$

where $\overline{\mathcal{M}}_{B-A}(H_{+\infty}, J)$ is understood as $\overline{\mathcal{M}}_{B-A}(H, J)$ in (4.11) with D replaced by $B-A$, and $[0, +\infty] = [0, +\infty) \cup \{+\infty\}$ is the compactification of $[0, +\infty)$ equipped with the structure of a bounded manifold obtained by requiring that

$$h : [0, +\infty] \to [0, 1],\ t \mapsto t/\sqrt{1+t^2}$$

is a diffeomorphism. Then by the standard arguments we can prove that the space in (4.26) is compact and Hausdorff with respect to the weak C^∞-topology. In addition, one has the obvious continuous evaluation map

$$\Xi_{B-A} : \overline{\mathcal{M}}_{B-A}(\{H_\rho\}, J) \to M \times M \tag{4.27}$$

given by $\Xi_{B-A}(\rho, \langle f\rangle) = (f(z_-), f(z_+))$ for $\langle f\rangle \in \overline{\mathcal{M}}_{B-A}(H_\rho, J)$ with $\rho \in [0, +\infty)$, and $\Xi_{B-A}(+\infty, \langle f\rangle) = (f(z_-), f(z_+))$ for $\langle f\rangle \in \overline{\mathcal{M}}_{B-A}(H_{+\infty}, J)$.

As above we can construct an associated virtual moduli cycle $\overline{\mathcal{M}}^\nu_{B-A}(\{H_\rho\}, J)$ of dimension $2n + 2c_1(B-A) + 1$ with the boundary

$$\partial\overline{\mathcal{M}}^\nu_{B-A}(\{H_\rho\}, J) = (-\overline{\mathcal{M}}^{\nu_0}_{B-A}(H_0, J)) \cup \overline{\mathcal{M}}^{\nu_{+\infty}}_{B-A}(H_{+\infty}, J).$$

Moreover, the evaluation map in (4.27) can naturally be extended onto the virtual module cycle, denoted by

$$\Xi^\nu_{B-A} : \overline{\mathcal{M}}^\nu_{B-A}(\{H_\rho\}, J) \to M \times M.$$

Let $n^{\nu_0}_{B-A}(a, b, h, g; H_0, J)$ (resp. $n^{\nu_{+\infty}}_{B-A}(a, b, h, g; H_{+\infty}, J)$) denote the intersection number of $\bar{E}^u_a \times \bar{E}^s_b$ and the restriction of Ξ^ν_{B-A} to $\overline{\mathcal{M}}^{\nu_0}_{B-A}(H_0, J)$ (resp. $\overline{\mathcal{M}}^{\nu_{+\infty}}_{B-A}(H_{+\infty}, J)$). Note that $\overline{\mathcal{M}}^{\nu_{+\infty}}_{B-A}(H_{+\infty}, J)$ is just a virtual moduli cycle associated with the space $\overline{\mathcal{M}}_{B-A}(H, J)$ defined in (4.12). By (4.18) we get

$$\Psi \circ \Phi(\langle a, A\rangle) = \sum_{\mu(\langle b,B\rangle)=\mu(\langle a,A\rangle)} n^{\nu_{+\infty}}_{B-A}(a, b, h, g; H_{+\infty}, J) \cdot \langle b, B\rangle. \tag{4.28}$$

As in [F, SZ] and [S2] we wish to define a homomorphism $\varphi : QC_*(M, \omega; h, g; \mathbb{Q}) \to QC_*(M, \omega; h, g; \mathbb{Q})$ such that

$$\Psi \circ \Phi - (\Psi \circ \Phi)_0 = \partial^Q\varphi + \varphi\partial^Q. \tag{4.29}$$

For $h \in \mathcal{O}(h_0)$, $\langle a, A\rangle \in (\mathrm{Crit}(h) \times \Gamma)_k$ and $\langle d, D\rangle \in (\mathrm{Crit}(h) \times \Gamma)_{k+1}$, the equality $\mu(a) - \mu(d) + 2c_1(D-A) + 1 = 0$ implies that for a generic $(h, g) \in \mathcal{O}(h_0) \times \mathcal{R}$ the evaluations $\bar{E}^u_a \times \bar{E}^s_d$ and Ξ^ν_{D-A} are intersecting transversally. So the rational intersection number

$$n_{D-A}(a, d, h, g; \{H_\rho\}, J) := (\bar{E}^u_a \times \bar{E}^s_d) \cdot \Xi^\nu_{D-A}$$

is well-defined. Then it is not difficult to check that φ defined by

$$\varphi(\langle a, A\rangle) = \sum_{\mu(\langle d,D\rangle)=\mu(\langle a,A\rangle)+1} n_{D-A}(a, d, h, g; \{H_\rho\}, J) \cdot \langle d, D\rangle \tag{4.30}$$

is an endomorphism of $QC_*(M, \omega; h, g; \mathbb{Q})$ and satisfies (4.29). That is,

$$\Psi \circ \Phi(\langle a, A\rangle) - (\Psi \circ \Phi)_0(\langle a, A\rangle) = \partial^Q\varphi(\langle a, A\rangle) + \varphi\partial^Q(\langle a, A\rangle)$$

for each $\langle a, A\rangle \in \mathrm{Crit}(h)\times\Gamma$. In fact, by the direct computation it suffice to prove

$$\begin{aligned} & n^{\nu_{+\infty}}_{B-A}(a,b,h,g;H_{+\infty},J) - n^{\nu_0}_{B-A}(a,b,h,g;H_0,J) \\ & = \textstyle\sum_{\mu(c)=\mu(b)+1} n_{B-A}(a,c,h,g;\{H_\rho\},J)n(c,b) \\ & - \textstyle\sum_{\mu(d)=\mu(a)-1} n(a,d)n_{B-A}(d,b,h,g;\{H_\rho\},J) \end{aligned} \tag{4.31}$$

for each $\langle b, B\rangle \in \mathrm{Crit}(h)\times\Gamma$ with $\mu(\langle b,B\rangle) = \mu(\langle a,A\rangle)$. To prove it we take a generic $(h,g)\in\mathcal{O}(h_0)\times\mathcal{R}$ so that the evaluations $\bar{E}^u_a\times\bar{E}^s_b$, $\bar{E}^u_a\times\bar{E}^s_c$ and $\bar{E}^u_d\times\bar{E}^s_b$ are transversal to Ξ^ν_{B-A}. By Lemmas 3.1 and 3.2, for a generic (h,g) the fibre product

$$(\overline{W}^u(a,h,g)\times\overline{W}^s(b,h,g))\times_{\bar{E}^u_a\times\bar{E}^s_b=\Xi^\nu_{B-A}}\overline{\mathcal{M}}^\nu_{B-A}(\{H_\rho\},J)$$

is a collection of compatible local cornered smooth manifolds of dimension 1 and with the natural orientations. Its boundary is given by

$$\begin{aligned} & (-(\overline{W}^u(a,h,g)\times\overline{W}^s(b,h,g))\times_{\bar{E}^u_a\times\bar{E}^s_b=\mathrm{E}^{\nu_0}_{B-A}}\overline{\mathcal{M}}^{\nu_0}_{B-A}(H_0,J)) \\ & \bigcup((\overline{W}^u(a,h,g)\times\overline{W}^s(b,h,g))\times_{\bar{E}^u_a\times\bar{E}^s_b=\mathrm{EV}^\nu_{B-A}}\overline{\mathcal{M}}^{\nu_{+\infty}}_{B-A}(H_{+\infty},J)) \\ & \bigcup\textstyle\bigcup_{\mu(d)=\mu(a)-1} n(a,d)\cdot((\overline{W}^u(a,h,g)\times W^s(d,h,g))\times_{R_{ad}}\overline{\mathcal{M}}^\nu_{B-A}(\{H_\rho\},J)) \\ & \bigcup\textstyle\bigcup_{\mu(c)=\mu(b)+1} n(c,b)\cdot(-(W^u(c,h,g)\times\overline{W}^s(b,h,g))\times_{R_{cb}}\overline{\mathcal{M}}^\nu_{B-A}(\{H_\rho\},J)). \end{aligned}$$

Here $R_{ad} = \bar{E}^u_a\times\bar{E}^s_d = \Xi^\nu_{B-A}$ and $R_{cb} = \bar{E}^u_c\times\bar{E}^s_b = \Xi^\nu_{B-A}$. This implies (4.31). To sum up we have proved:

Proposition 4.6. $\Psi\circ\Phi$ *is chain homotopy equivalent to* $(\Psi\circ\Phi)_0$.

Step 4. We need to make further homotopy. For $\tau\in[0,1]$ we define

$$\bar{\partial}_{J,\tau H_0}u(s,t) = \partial_s u(s,t) + J(u)(\partial_t u - \tau(\beta_+(s+1)\cdot\beta_+(1-s))X_H(t,u)) = 0.$$

In Definition 4.4 we replace $\bar{\partial}_{J,H_\rho}$ with $\bar{\partial}_{J,\tau H_0}$ and define the corresponding stable $(J,\tau H_0)$-map of class D. Let $\overline{\mathcal{M}}_D(\tau H_0,J)$ be the space of all equivalence classes of such maps. For $\langle f\rangle\in\overline{\mathcal{M}}_{B-A}(\tau H_0,J)$, as in (4.24) we can estimate

$$E^\tau(f) := \textstyle\int_0^1|\partial_s f^P|^2_{g_J}dsdt + \sum_i\int_{B_i}(f^B_i)^*\omega \le \omega(B-A) + 2\tau\max|H|.$$

As above we can construct a virtual moduli cycle $\overline{\mathcal{M}}^\nu_{B-A}(\{\tau H_0\},J)$ of the compact space $\cup_{\tau\in[0,1]}\{\tau\}\times\overline{\mathcal{M}}_{B-A}(\tau H_0,J)$ of dimension $2n+2c_1(B-A)+1$ and with boundary

$$\partial\overline{\mathcal{M}}^\nu_{B-A}(\{\tau H_0\},J) = (-\overline{\mathcal{M}}^{\nu^0}_{B-A}(0,J))\cup\overline{\mathcal{M}}^{\nu^1}_{B-A}(H_0,J).$$

(Actually, $\overline{\mathcal{M}}^{\nu^1}_{B-A}(H_0,J)$ can be chosen as $\overline{\mathcal{M}}^{\nu_0}_{B-A}(H_0,J)$.) For $\tau = 0,1$ and a generic $(h,g)\in\mathcal{O}(h_0)\times\mathcal{R}$, using the evaluation map

$$\mathrm{E}^{\nu^\tau}_{B-A}:\overline{\mathcal{M}}^{\nu^\tau}_{B-A}(\tau H_0,J)\to M\times M,\ f\mapsto(f(z_-),f(z_+)),$$

we get the well-defined rational intersection number

$$n^{\nu^\tau}_{B-A}(a,b,h,g;\tau H_0,J):=(\bar E^u_a\times\bar E^s_b)\cdot \mathrm{E}^{\nu^\tau}_{B-A}.$$

By (4.25) we have

$$n^{\nu^1}_{B-A}(a,b,h,g;H_0,J)=n^{\nu_0}_{B-A}(a,b,h,g;H_0,J)$$

since they are independent of the generic choices of ν and (g,h). Therefore, as in Step 3 we can easily prove that $(\Psi\circ\Phi)_0$ and thus $\Phi\circ\Phi$ are chain homotopy equivalent to $(\Psi\circ\Phi)^0$ defined by

$$(\Psi\circ\Phi)^0(\langle a,A\rangle)=\sum_{\mu(\langle b,B\rangle)=\mu(\langle a,A\rangle)} n^{\nu^0}_{B-A}(a,b,h,g;0,J)\cdot\langle b,B\rangle. \tag{4.32}$$

Here as in Proposition 4.5 it can be proved that $(\Psi\circ\Phi)^0$ is a chain homomorphism. We omit it. Now Theorem 4.1 can follow from this and the following result:

Proposition 4.7. *The numbers $n^{\nu^0}_{B-A}(a,b,h,g;0,J)$ at (4.32) satisfies*

$$n^{\nu^0}_{B-A}(a,b,h,g;0,J)=\begin{cases}1 & \text{if } a=b \text{ and } A=B,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. *Case 1:* $A\ne B$. In this case $\overline{\mathcal M}_{B-A}(0,J)$ contains no the constant maps. Moreover, the domain of elements of $\overline{\mathcal M}_{B-A}(0,J)$ is only 0-pointed semistable $\mathcal F$-curves with at least a principal component and the operator $\bar\partial_J$ is invariant under action of the automorphism group of the domain of a semistable $\mathcal F$-curve. Thus as in [LiuT2] the associated virtual moduli cycle $\overline{\mathcal M}^{\nu^0}_{B-A}(0,J)$ can be required to carry a free $\mathbb S^1$-action under which the evaluation is invariant. Hence for a generic $(h,g)\in\mathcal O(h_0)\times\mathcal R$ the fibre product

$$(\overline{W}^u(a,h,g)\times\overline{W}^s(b,h,g))\times_{(\bar E^u_a\times\bar E^s_b)=\mathrm{E}^{\nu^0}_{B-A}}\overline{\mathcal M}^{\nu^0}_{B-A}(0,J)$$

must be empty under the condition (4.2). We get $n^{\nu^0}_{B-A}(a,b,h,g;0,J)=0$.
Case 2: $A=B$. Now $\overline{\mathcal M}_{B-A}(0,J)$ can naturally be identified with M and thus $\overline{\mathcal M}^{\nu^0}_{B-A}(0,J)$ may be taken as $\overline{\mathcal M}_{B-A}(0,J)$. Note that $\mu(a)=\mu(b)$ in the present case and that $\overline{W}^u(a,h,g)\cap\overline{W}^s(b,h,g)$ carries a free $\mathbb R$-action if it is nonempty. The conclusions follow naturally. □

Summing up the above arguments we complete the proof of Theorem 4.1.

Remark 4.8. If the Morse function h has only critical points of even index then Φ is a right inverse of Ψ as the chain homomorphisms between $QC_*(M,\omega;h,g;\mathbb Q)$ and $C_*(H,J,\nu;\mathbb Q)$. Indeed, carefully checking the proof of Th.4.1 one will find that $n^{\nu_{+\infty}}_{B-A}(a,b,h,g;H_{+\infty},J)-n^{\nu_0}_{B-A}(a,b,h,g;H_0,J)=0$ in (4.31). The same reasoning yields $n^{\nu^1}_{B-A}(a,b,h,g;H_0,J)=n^{\nu^0}_{B-A}(a,b,h,g;0,J)$. By Proposition 4.7 and (4.28) we get that $\Psi\circ\Phi(\langle a,A\rangle)=\langle a,A\rangle$ for each $\langle a,A\rangle$.

Theorem 4.9. *$\Phi\circ\Psi$ is chain homotopy equivalent to the identity. Consequently, Φ induces a surjective Λ_ω-module homomorphism from $QH_*(h,g;\mathbb{Q})$ to $HF_*(M,\omega;H,J,\nu;\mathbb{Q})$.*

Proof. The proof is similar to that of Theorem 4.1. We only give main steps.
Step 1. For every $\tilde{x}\in\tilde{\mathcal{P}}_k(H)$ the direct computation gives

$$\Phi\circ\Psi(\tilde{x})=\sum_{\mu(\tilde{y})=k}m^{\nu}_{-,+}(\tilde{x},\tilde{y})\cdot\tilde{y},$$
$$m^{\nu}_{-,+}(\tilde{x},\tilde{y})=\sum_{\mu(\langle a,A\rangle)=k}n^{\nu^-}_{-}(a,\tilde{x}\sharp(-A))\cdot n^{\nu^+}_{+}(a,\tilde{y}\sharp(-A)).$$

As in the proof of Theorem 4.1 using Proposition 3.4 we can easily prove the second sum to be finite. That is, there are only finitely many $\langle a,A\rangle\in(\mathrm{Crit}(h)\times\Gamma)_k$ such that $n^{\nu^-}_{-}(a,\tilde{x}\sharp(-A))\cdot n^{\nu^+}_{+}(a,\tilde{y}\sharp(-A))\neq 0$. Actually, there exist finitely many $A\in\Gamma$, saying $A_1,\cdots,A_s$, such that

$$\overline{\mathcal{M}}_-(\tilde{x}\sharp(-A_i);H,J)\neq\emptyset\quad\text{and}\quad\overline{\mathcal{M}}_+(\tilde{y}\sharp(-A_i);H,J)\neq\emptyset$$

for $i=1,\cdots,s$. Note that the product $n^{\nu^-}_{-}(a,\tilde{x}\sharp(-A))\cdot n^{\nu^+}_{+}(a,\tilde{y}\sharp(-A))$ can be explained as the intersection number of the product evaluations

$$\mathrm{EV}^{\nu^-}_{-}\times\mathrm{EV}^{\nu^+}_{+}:\overline{\mathcal{M}}^{\nu^-}_{-}(\tilde{x}\sharp(-A);H,J)\times\overline{\mathcal{M}}^{\nu^+}_{+}(\tilde{y}\sharp(-A);H,J)\to M\times M$$

and $\bar{E}^s_a\times\bar{E}^u_a:\overline{W}^s(a,h,g)\times\overline{W}^u(a,h,g)\to M\times M$,

$$(\bar{E}^s_a\times\bar{E}^u_a)\cdot(\mathrm{EV}^{\nu^-}_{-}(\tilde{x}\sharp(-A))\times\mathrm{EV}^{\nu^+}_{+}(\tilde{y}\sharp(-A)))$$

for a generic $(h,g)\in\mathcal{O}(h_0)\times\mathcal{R}$, and that the fibre product

$$(\overline{W}^s(a,h,g)\times\overline{W}^u(a,h,g))\times_R(\overline{\mathcal{M}}^{\nu^-}_{-}(\tilde{x}\sharp(-A);H,J)\times\overline{\mathcal{M}}^{\nu^+}_{+}(\tilde{y}\sharp(-A);H,J))$$

is an empty set for a generic $(h,g)\in\mathcal{O}(h_0)\times\mathcal{R}$ even if $\mu(\langle a,A\rangle)\neq k=\mu(\tilde{x})=\mu(\tilde{y})$. Here $R=\bar{E}^s_a\times\bar{E}^u_a=\mathrm{EV}^{\nu^-}_{-}\times\mathrm{EV}^{\nu^+}_{+}$. Thus

$$(4.33)\quad\begin{cases}m^{\nu}_{-,+}(\tilde{x},\tilde{y})\\ =\sum_{\langle a,A\rangle\in\mathrm{Crit}(h)\times\Gamma}(\bar{E}^s_a\times\bar{E}^u_a)\cdot(\mathrm{EV}^{\nu^-}_{-}(\tilde{x}\sharp(-A))\times\mathrm{EV}^{\nu^+}_{+}(\tilde{y}\sharp(-A)))\\ =\sum_{a\in\mathrm{Crit}(h)}\sum_{i=1}^s(\bar{E}^s_a\times\bar{E}^u_a)\cdot(\mathrm{EV}^{\nu^-}_{-}(\tilde{x}\sharp(-A_i))\times\mathrm{EV}^{\nu^+}_{+}(\tilde{y}\sharp(-A_i))).\end{cases}$$

Step 2. To understand this sum, for $(h,g)\in\mathcal{O}(h_0)\times\mathcal{R}$ and $\rho\geq 0$ we denote by

$$\mathcal{M}_\rho(h,g):=\{\gamma\in C^\infty([-\rho,\rho],M)\,|\,\dot{\gamma}+\nabla_g h(\gamma)=0\}$$

It is a compact manifold of dimension $2n$ and $\mathcal{M}_0(h,g)$ can naturally be identified with M. Using the gluing techniques in the Morse homology (cf. Theorem 6.8 in [Lu2]), the natural weak compactification of the noncompact manifold $\cup_{\rho\geq 0}\mathcal{M}_\rho(h,g)$ of dimension $2n+1$ is given by

$$\overline{\cup_{\rho\geq 0}\mathcal{M}_\rho(h,g)}:=\cup_{\rho=0}^{\infty}\mathcal{M}_\rho(h,g),$$

where

$$\mathcal{M}_\infty(h,g):=\cup_{a\in\mathrm{Crit}(h)}\overline{W}^s(a,h,g)\times\overline{W}^u(a,h,g)$$

and the weak convergence of a sequence $\{\gamma_m\}\subset\mathcal{M}_{\rho_m}(h,g)$, $\rho_m\to\infty$, towards a pair $(u,v)\in\mathcal{M}_\infty(h,g)$ is understood in an obvious way (cf.[AuB],[Sch1]and [Sch4]). The space has the structure of a manifold with corners and with boundary

$$\partial\overline{\cup_{\rho\ge 0}\mathcal{M}_\rho(h,g)}=(-\mathcal{M}_0(h,g))\cup(\cup_{a\in\mathrm{Crit}(h)}\overline{W}^s(a,h,g)\times\overline{W}^u(a,h,g)).$$

Moreover, there exists a smooth evaluation

$$\mathrm{ev}_{h,g}:\overline{\cup_{\rho\ge 0}\mathcal{M}_\rho(h,g)}\to M\times M$$

given by $\mathrm{ev}_{h,g}(\gamma)=(\gamma(-\rho),\gamma(\rho))$ for $\gamma\in\mathcal{M}_\rho(h,g)$ such that

$$\mathrm{ev}_{h,g}|_{\overline{W}^s(a,h,g)\times\overline{W}^u(a,h,g)}=\bar{E}^s_a\times\bar{E}^u_a\ \text{ for each } a\in\mathrm{Crit}(h).$$

We also denote by $\mathrm{ev}^\rho_{h,g}$ the restriction of $\mathrm{ev}_{h,g}$ to $\mathcal{M}_\rho(h,g)$ for $0\le\rho\le\infty$. To make further arguments we need to assume that

$$\mathrm{EV}^{\nu^-}_-(\tilde{x}\sharp(-A_i))\pitchfork\mathrm{EV}^{\nu^+}_+(\tilde{y}\sharp(-A_i)),\ i=1,\cdots,s. \tag{4.34}$$

These may actually be obtained for a generic small $(\nu^-,\nu^+)\in R^-_\varepsilon\times R^+_\varepsilon$ by increasing some points $f^-_j\in\overline{\mathcal{M}}_-(\tilde{x}\sharp(-A_i);H,J)$ and $f^+_j\in\overline{\mathcal{M}}_+(\tilde{y}\sharp(-A_i);H,J)$ and enlarging the spaces $R^\pm_\varepsilon$ in the construction of the virtual module cycles.

Using these, for a generic pair $(h,g)\in\mathcal{O}(h_0)\times\mathcal{R}$ the fibre product

$$\overline{\cup_{\rho\ge 0}\mathcal{M}_\rho(h,g)}\times_{R_3}(\cup_{i=1}^s\overline{\mathcal{M}}^{\nu^-}_-(\tilde{x}\sharp(-A_i);H,J)\times\overline{\mathcal{M}}^{\nu^+}_+(\tilde{y},w\sharp(-A_i);H,J))$$

is a collection of compatible local cornered smooth manifolds of dimension 1 and with the natural orientations. Here R_3 represents $ev(h,g)=\mathrm{EV}^{\nu^-}_-\times\mathrm{EV}^{\nu^+}_+$. Its boundary is the union of the following four sets

$$-\mathcal{M}_0(h,g)\times_{R_1}(\cup_{i=1}^s\overline{\mathcal{M}}^{\nu^-}_-(\tilde{x}\sharp(-A_i))\times\overline{\mathcal{M}}^{\nu^+}_+(\tilde{y}\sharp(-A_i))),$$

$$(\bigcup_{a\in\mathrm{Crit}(h)}\overline{W}^s(a,h,g)\times\overline{W}^u(a,h,g))\times_{R_2}(\cup_{i=1}^s\overline{\mathcal{M}}^{\nu^-}_-(\tilde{x}\sharp(-A_i))\times\overline{\mathcal{M}}^{\nu^+}_+(\tilde{y}\sharp(-A_i))),$$

$$\bigcup_{\mu(\tilde{z})=\mu(\tilde{x})-1}-m(\tilde{x},\tilde{z})\cdot\overline{\bigcup_{\rho\ge 0}\mathcal{M}_\rho(h,g)}\times_{R_3}\Big(\cup_{i=1}^s\overline{\mathcal{M}}^{\nu^-}_-(\tilde{z}\sharp(-A_i))\times\overline{\mathcal{M}}^{\nu^+}_+(\tilde{y}\sharp(-A_i))\Big),$$

$$\bigcup_{\mu(\tilde{z}')=\mu(\tilde{y})+1}m(\tilde{z}',\tilde{y})\cdot\overline{\bigcup_{\rho\ge 0}\mathcal{M}_\rho(h,g)}\times_{R_3}\Big(\cup_{i=1}^s\overline{\mathcal{M}}^{\nu^-}_-(\tilde{x}\sharp(-A_i))\times\overline{\mathcal{M}}^{\nu^+}_+(\tilde{z}'\sharp(-A_i))\Big).$$

Here R_1 and R_2 represent $\mathrm{ev}^0_{h,g}=\mathrm{EV}^{\nu^-}_-\times\mathrm{EV}^{\nu^+}_+$ and $\mathrm{ev}^\infty_{h,g}=\mathrm{EV}^{\nu^-}_-\times\mathrm{EV}^{\nu^+}_+$ respectively, $m(\tilde{x},\tilde{z})=\sharp(C(\overline{\mathcal{M}}^\nu(\tilde{x},\tilde{z})))$ and $m(\tilde{z}',\tilde{y})=\sharp(C(\overline{\mathcal{M}}^\nu(\tilde{z}',\tilde{y})))$. Hereafter we omit H,J in $\overline{\mathcal{M}}^{\nu^\pm}_\pm(\cdot;H,J)$ without confusions. Note that

$$\sharp\Big(\Big(\bigcup_{a\in\mathrm{Crit}(h)}\overline{W}^s(a,h,g)\times\overline{W}^u(a,h,g)\Big)\times_{\mathcal{R}_2}\Big(\bigcup_{i=1}^s\overline{\mathcal{M}}^{\nu^-}_-(\tilde{x}\sharp(-A_i))\times\overline{\mathcal{M}}^{\nu^+}_+(\tilde{y}\sharp(-A_i))\Big)\Big)$$

is exactly the number in (4.33), and that because of (4.34),

$$\sharp(\mathcal{M}_0(h,g)\times_{\mathcal{R}_1}(\cup_{i=1}^s\overline{\mathcal{M}}_-^{\nu^-}(\tilde{x}\sharp(-A_i))\times\overline{\mathcal{M}}_+^{\nu^+}(\tilde{y}\sharp(-A_i))))$$

is exactly the sum of the intersection numbers

$$\sum_{i=1}^s \mathrm{EV}_-^{\nu^-}(\tilde{x}\sharp(-A_i))\cdot\mathrm{EV}_+^{\nu^+}(\tilde{y}\sharp(-A_i)) \tag{4.35}$$

As in the proof of Theorem 4.1 the above boundary relations lead to:

Proposition 4.10. $\Phi\circ\Psi$ *is chain homotopy equivalent to the homomorphism defined by*

$$\overline{\Phi\circ\Psi}(\tilde{x})=\sum_{\mu(\tilde{y})=\mu(\tilde{x})}(\sum_{i=1}^s\mathrm{EV}_-^{\nu^-}(\tilde{x}\sharp(-A_i))\cdot\mathrm{EV}_+^{\nu^+}(\tilde{y}\sharp(-A_i)))\cdot\tilde{y}. \tag{4.36}$$

Now we also need to prove that $\overline{\Phi\circ\Psi}$ is a chain homomorphism from $C_*(H,J;\mathbb{Q})$ to itself yet. The ideas are same as those of Proposition 4.5. Let us outline it as follows. Firstly, (2.6) implies that if the sum in (4.35) is not zero then

$$\mathcal{F}_H(\tilde{y})\le\max|H|-\min_{1\le i\le s}\omega(A_i)\ \text{ and }\ \mathcal{F}_H(\tilde{x})\ge-\max|H|-\max_{1\le i\le s}\omega(A_i).$$

From these it easily follows that $\overline{\Phi\circ\Psi}$ maps $C_*(H,J;\mathbb{Q})$ to itself. Next, by the direct computation we easily reduce the proof of $\overline{\Phi\circ\Psi}\circ\partial^F=\partial^F\circ\overline{\Phi\circ\Psi}$ to proving that for given $\tilde{x}\in\tilde{\mathcal{P}}_k(H)$ and $\tilde{z}\in\tilde{\mathcal{P}}_{k-1}(H)$ the following holds.

$$\begin{aligned}&\sum_{i=1}^s\sum_{\mu(\tilde{y})=k}(\mathrm{EV}_-^{\nu^-}(\tilde{x}\sharp(-A_i))\cdot\mathrm{EV}_+^{\nu^+}(\tilde{y}\sharp(-A_i)))\cdot n_i(\tilde{y},\tilde{z})\\ &=\sum_{i=1}^s\sum_{\mu(\tilde{z}')=k-1}n_i(\tilde{x},\tilde{z}')\cdot(\mathrm{EV}_-^{\nu^-}(\tilde{z}'\sharp(-A_i))\cdot\mathrm{EV}_+^{\nu^+}(\tilde{z}\sharp(-A_i))).\end{aligned}$$

Here and in the following unions $n_i(\tilde{x},\tilde{z}')=\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{x}\sharp(-A_i),\tilde{z}'\sharp(-A_i))))$ and $n_i(\tilde{y},\tilde{z})=\sharp(C(\overline{\mathcal{M}}^{\nu}(\tilde{y}\sharp(-A_i),\tilde{z}\sharp(-A_i))))$. In fact, by (4.34) the fibre product

$$\overline{\mathcal{M}}_-^{\nu^-}(\tilde{x}\sharp(-A_i))\times_{\mathrm{EV}_-^{\nu^-}=\mathrm{EV}_+^{\nu^+}}\overline{\mathcal{M}}_+^{\nu^+}(\tilde{z}\sharp(-A_i))$$

is a collection of compatible local cornered smooth manifolds of dimension 1 and with the natural orientations. Its boundary is given by

$$\begin{aligned}&(\cup_{\mu(\tilde{z}')=k-1}n_i(\tilde{x},\tilde{z}')\cdot(\overline{\mathcal{M}}_-^{\nu^-}(\tilde{z}',\sharp(-A_i))\times_{\mathrm{EV}_-^{\nu^-}=\mathrm{EV}_+^{\nu^+}}\overline{\mathcal{M}}_+^{\nu^+}(\tilde{z}\sharp(-A_i))))\\ &\cup(-\cup_{\mu(\tilde{y})=k}(\overline{\mathcal{M}}_-^{\nu^-}(\tilde{x}\sharp(-A_i))\times_{\mathrm{EV}_-^{\nu^-}=\mathrm{EV}_+^{\nu^+}}\overline{\mathcal{M}}_+^{\nu^+}(\tilde{y}\sharp(-A_i)))\cdot n_i(\tilde{y},\tilde{z})).\end{aligned}$$

Notice that the sum in (4.35) is exactly

$$\sharp(\cup_{i=1}^s\overline{\mathcal{M}}_-^{\nu^-}(\tilde{x}\sharp(-A_i))\times_{\mathrm{EV}_-^{\nu^-}=\mathrm{EV}_+^{\nu^+}}\overline{\mathcal{M}}_+^{\nu^+}(\tilde{y}\sharp(-A_i))).$$

They together lead to the conclusions.

Step 3. To understand the number in (4.35) we introduce:

Definition 4.11. Given $\tilde{x}, \tilde{y} \in \tilde{\mathcal{P}}(H)$ and a semistable $\mathcal{F}$-curve $(\Sigma, \underline{l})$ with at least two principal components (cf. Def.2.1), a continuous map

$$f : \Sigma \setminus \{z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_{N_p+1}\} \to M$$

is called a **stable** (J, H)**-broken trajectory** if we divide $(\Sigma, \underline{l})$ into two semistable $\mathcal{F}$-curve $(\Sigma_-, \underline{l}^-)$ and $(\Sigma_+, \underline{l}^+)$ at some double point z_i between two principal components, $1 < i < N_p + 1$, then $f|_{\Sigma_-}$ and $f|_{\Sigma_+}$ are the stable $(J, H)_-$-disk with cap $\tilde{x}$ and stable $(J, H)_+$-disk with cap $\tilde{y}$ respectively.

We can also define its equivalence class in an obvious way. Denote the space of all such equivalence classes by $\overline{\mathcal{M}}_{-+}(\tilde{x}, \tilde{y}; H, J)$. Let $\mathcal{M}_{-+}(\tilde{x}, \tilde{y}; H, J)$ be its subspace consisting of those elements whose domains have only two principal components and have no any bubble components. Then the former is the natural compactification of the latter, and has the virtual dimension $\mu(\tilde{x}) - \mu(\tilde{y})$. We can, as before, construct an associated virtual module cycle $\overline{\mathcal{M}}^{\nu}_{-+}(\tilde{x}, \tilde{y}; H, J)$ of dimension $\mu(\tilde{x}) - \mu(\tilde{y})$. Specially, if $\mu(\tilde{x}) = \mu(\tilde{y})$ this virtual moduli cycle determines a well-defined rational number $n^{\nu}_{-+}(\tilde{x}, \tilde{y}; H, J)$ which is independent of a generic choice of ν in the obvious way. As in the proof of Theorem 4.1, by carefully checking the construction of the virtual module cycle we can prove that

$$n^{\nu}_{-+}(\tilde{x}, \tilde{y}; H, J) = \mathrm{EV}^{\nu^-}_{-}(\tilde{x}) \cdot \mathrm{EV}^{\nu^+}_{+}(\tilde{y}).$$

Moreover, as showed in Step 1 there exist only finitely many $A_i \in \Gamma$ such that $\overline{\mathcal{M}}_{-+}(\tilde{x}\sharp(-A_i), \tilde{y}\sharp(-A_i); H, J) \neq \emptyset$, $i = 1, \cdots, s$. So the virtual module cycle associated with $\cup_{i=1}^{s}\overline{\mathcal{M}}_{-+}(\tilde{x}\sharp(-A_i), \tilde{y}\sharp(-A_i); H, J)$ can be taken as

$$\cup_{i=1}^{s}\overline{\mathcal{M}}^{\nu_i}_{-+}(\tilde{x}\sharp(-A_i), \tilde{y}\sharp(-A_i); H, J).$$

It follows that

$$\sharp(\cup_{i=1}^{s}\overline{\mathcal{M}}_{-+}(\tilde{x}\sharp(-A_i), \tilde{y}\sharp(-A_i); H, J))^{\nu} = \sum_{i=1}^{s} n^{\nu}_{-+}(\tilde{x}\sharp(-A_i), \tilde{y}\sharp(-A_i); H, J).$$

Thus (4.36) becomes

$$\overline{\Phi \circ \Psi}(\tilde{x}) = \sum_{\mu(\tilde{y})=\mu(\tilde{x})} \sharp(\cup_{i=1}^{s}\overline{\mathcal{M}}_{-+}(\tilde{x}\sharp(-A_i), \tilde{y}\sharp(-A_i); H, J))^{\nu} \cdot \tilde{y}. \tag{4.37}$$

Step 4. Furthermore, for $\rho \geq 0$ and $f : \mathbb{R} \times S^1 \to M$ we define $\bar{\partial}_{J,H^\rho} f$ by

$$\bar{\partial}_{J,H^\rho} f(s,t) = \partial_s f(s,t) + J(f)(\partial_t f - (\beta_+(s-\rho) + \beta_+(-s-\rho))X_H(t,f)) = 0,$$

where β_+ is as in (4.22). For such a map f we still define the energy of it by (4.23).

Definition 4.12. Given $[x, v], [y, w] \in \tilde{\mathcal{P}}(H)$ and a semistable $\mathcal{F}$-curve $(\Sigma, \underline{l})$ as in Definition 2.1, a continuous map $f : \Sigma \setminus \{z_1, \cdots, z_{N_p+1}\} \to M$ is called a **stable** $(J, H)^\rho$**-trajectory** if there exist $[x, v] = [x_1, u_1], \cdots, [x_{N_p+1}, u_{N_p+1}] =$

$[y, w]$ such that (1), (2), (3) in Def.2.1 are satisfied unless (i) in Def.2.1(1) is replaced by $\bar{\partial}_{J,H^\rho} f_k^P = 0$ and $E_\rho(f_k^P) < +\infty$ in some principal component P_k.

Let $\overline{\mathcal{M}}(\tilde{x}, \tilde{y}; J, H^\rho)$ denote the space of equivalence classes of all stable $(J, H)^\rho$-trajectories from $\tilde{x}$ to $\tilde{y}$. As in §2 we can prove that this space is compact according to the weak C^∞-topology and construct the corresponding virtual moduli cycle $\overline{\mathcal{M}}^\nu(\tilde{x}, \tilde{y}; J, H^\rho)$ of dimension $\mu(\tilde{x}) - \mu(\tilde{y})$. Specially, if $\mu(\tilde{x}) = \mu(\tilde{y})$ we can associate a rational number to it, denoted by $m^\nu(\tilde{x}, \tilde{y}; J, H^\rho)$. Consider the space $\cup_{\rho \geq 0} \overline{\mathcal{M}}(\tilde{x}, \tilde{y}; J, H^\rho)$. As before one easily shows that the natural weak compactification of it is given by

$$(\cup_{\rho\geq 0}\overline{\mathcal{M}}(\tilde{x}, \tilde{y}; J, H^\rho)) \cup (\cup_{i=1}^s \overline{\mathcal{M}}_{-+}(\tilde{x}\sharp(-A_i), \tilde{y}\sharp(-A_i); H, J)).$$

Using it we can construct a virtual module cycle of dimension 1 and prove:

Proposition 4.13. $\overline{\Phi \circ \Psi}$ *in (4.37) is chain homotopy equivalent to the homomorphism given by*

$$(\Phi \circ \Psi)^0(\tilde{x}) = \sum_{\mu(\tilde{y})=\mu(\tilde{x})} m^\nu(\tilde{x}, \tilde{y}; J, H^0) \cdot \tilde{y}.$$

Here we have assumed that $(\Phi \circ \Psi)^0$ is a chain homomorphism from $C_*(H, J; \mathbb{Q})$ to itself. It can be proved as above. We omit it. As in [F, SZ] and [LiuT1], by taking a regular homotopy from (J, H) to $(J, (\beta_+(\cdot) + \beta_+(-\cdot))H)$ we can prove that $(\Phi \circ \Psi)^0$ is chain homotopy equivalent to the homomorphism defined by

$$\overline{(\Phi \circ \Psi)}^0(\tilde{x}) = \sum_{\mu(\tilde{y})=\mu(\tilde{x})} \sharp(C(\overline{\mathcal{M}}^\nu(\tilde{x}, \tilde{y}; J, H)))) \cdot \tilde{y}.$$

Since $\mu(\tilde{y}) = \mu(\tilde{x})$ it is easily checked that $\sharp(C(\overline{\mathcal{M}}^\nu(\tilde{x}, \tilde{y}; J, H))) = 1$ as $\tilde{x} = \tilde{y}$, and $\sharp(C(\overline{\mathcal{M}}^\nu(\tilde{x}, \tilde{y}; J, H))) = 0$ otherwise. That is, $\overline{(\Phi \circ \Psi)}^0 = id$. This fact, Proposition 4.13, (4.37) and Proposition 4.10 together prove Theorem 4.9. □

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